A Functional Calculus for
\((\alpha, \alpha + 1) - type \ R \text{ Operators}\)

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Abstract

A closed densely defined operator \(H\), on a Banach space \(X\), whose spectrum is contained in \(\mathbb{R}\) and satisfies

\[
\|(z - H)^{-1}\| \leq c \frac{(3)^{\alpha}}{|z|^{-\beta}} \quad \forall \ z \notin \mathbb{R} \quad \text{with} \quad (3)^{\alpha} := \sqrt{|z|^2 + 1} \quad (1)
\]

for some \(\alpha, \beta \geq 0; c > 0\), is said to be of \((\alpha, \beta) - type \ \mathbb{R}\) (notation introduced in [10]). For \((\alpha, \alpha + 1) - type \ \mathbb{R}\) operators we constructed an \(\mathcal{A}\)-functional calculus in a more general Banach space setting (where \(\mathcal{A}\) is the algebra of smooth functions on \(\mathbb{R}\) that decay like \((\sqrt{1 + x^2})^\beta\) as \(|x| \to \infty\), for some \(\beta < 0\). This algebra is fully characterized in [9]).

We then show that our functional calculus coincides with \(C_0\)-functional calculus for an unbounded operator acting on a Hilbert space.

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1 The definition

Let $\mathfrak{A}$ be the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$. This algebra is fully characterized in [9]. Next let $H$ be $(\alpha, \alpha + 1) - \text{type } \mathbb{R}$ operator (introduced in [10]). The motivation for our definition of $f(H)$ comes from two ideas. Firstly, a version of Hörmander’s concept of almost analytic extensions [6, 7], as contained in the following definition.

**Definition 1.1** Given $f \in \mathfrak{A}$ and $n \geq 0$, an *almost analytic extension* of $f$ to $\mathbb{C}$ is

\[
\tilde{f}_{\varphi,n}(x, y) := \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \varphi(x, y)
\]

\[
:= \left\{ f(x) + \sum_{r=1}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right\} \varphi(x, y)
\]

where

\[
\varphi(x, y) = \tau \left( \frac{y}{|y|} \right) \quad \text{with} \quad \langle x \rangle := \sqrt{x^2 + 1}
\]

and $\tau$ is non-negative $C^\infty_c(\mathbb{R})$ function such that $\tau(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2 \end{cases}$.

The second idea in our definition of $f(H)$ comes from the Helffer and Sjöstrand [5] integral formula,

\[
f(H) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} (z - H)^{-1} dxdy
\]

for a suitable function and operator $H$.

**Lemma 1.2** Let $f \in \mathfrak{A}$, then $\left| \frac{\partial}{\partial \bar{z}} \tilde{f}_{\varphi,n}(x, y) \right| = O(|y|^n)$ as $|y| \to 0$ for a fixed $x$. Moreover we can find $c' \in \mathbb{R}$ such that

\[
\left| \frac{\partial}{\partial \bar{z}} \tilde{f}_{\varphi,n}(x, y) \right| \leq c' |y|^n \quad \text{as} \ z \to x \in \mathbb{R}.
\]
Proof.
\[
\frac{\partial}{\partial x}(\tilde{f}_{\varphi,n}(z)) = \sum_{r=0}^{n} \frac{f^{(r+1)}(x)(iy)^r}{r!} \varphi(x,y) + \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \varphi_x(x,y)
\]
and
\[
\frac{\partial}{\partial y}(\tilde{f}_{\varphi,n}(z)) = \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^{r-1}}{(r-1)!} \varphi(x,y) + \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \varphi_y(x,y)
\]
thus
\[
\frac{\partial}{\partial z}(\tilde{f}_{\varphi,n}(z)) = \frac{1}{2} \left( \frac{\partial \tilde{f}_{\varphi,n}}{\partial x} + i \frac{\partial \tilde{f}_{\varphi,n}}{\partial y} \right) (z)
\]
\[
= \frac{1}{2} \left( \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right) (\varphi_x + i \varphi_y)(z) + \frac{1}{2} f^{(n+1)}(x)(iy)^n \varphi(z)
\]
(5)

Now,
\[
\text{supp} (\varphi_x + i \varphi_y) \subseteq \left\{ (x, y) : 1 \leq \frac{|y|}{\langle x \rangle} \leq 2 \right\}
\]
\[
= \left\{ (x, y) : \langle x \rangle \leq |y| \leq 2 \langle x \rangle \right\}
\]
\[
\subseteq \left\{ (x, y) : 1 \leq |y| \leq 2 \langle x \rangle \right\}
\]
(6)

Therefore \(\varphi_x + i \varphi_y\) vanishes on the strip \(\Omega := \{(x, y) : -1 \leq y \leq 1\}\). So
\[
\left| \frac{\partial}{\partial z}(\tilde{f}_{\varphi,n}(x, y)) \right| = \frac{1}{2} f^{(n+1)}(x) \frac{|y|^n}{n!} \text{ for } (x, y) \in \Omega
\]
\[
= M_x |y|^n
\]

With \(M_x = \frac{\|f^{(n+1)}(x)\|}{2n!}\). Thus, \(\left| \frac{\partial}{\partial z} \tilde{f}_{\varphi,n}(x, y) \right| = O(|y|^n) \text{ as } |y| \to 0\) for a fixed \(x\).

Moreover, since \(f \in \mathfrak{A}\) we can find some \(\beta < 0\) and \(c' > 0\) such that
\[
M_x = \frac{|f^{(n+1)}(x)|}{2n!}
\]
\[
\leq c' \langle x \rangle^{-\beta - n - 1} \text{ for all } (x, y) \in \Omega
\]
\[
\leq c'
\]
\[
< \infty
\]
since \(\langle x \rangle^{-\beta - n - 1} \leq 1\) for all \(x \in \mathbb{R}\). Therefore \(\left| \frac{\partial}{\partial z}(\tilde{f}_{\varphi,n}(x, y)) \right| \leq c' |y|^n\) as \(z \to x \in \mathbb{R}\). \(\square\)

Lemma 1.3 If \(\varphi(x, y) := \tau \left( \frac{y}{|y|} \right) \) with \(\tau\), a non-negative \(C^\infty_c(\mathbb{R})\) function such that \(\tau(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2, \end{cases} \)

then \(\|\varphi_x + i \varphi_y\|(x, y) \leq \frac{K}{\langle x \rangle} \) for some \(K > 0\).
\textbf{Proof.}

$$|\varphi_x(x, y)| = \left| \frac{\partial}{\partial x} \tau \left( \frac{y}{\langle \phi \rangle} \right) \right|$$

$$= \left| \tau' \left( \frac{y}{\langle \phi \rangle} \right) \cdot \frac{\partial}{\partial x} (y \langle \phi \rangle^{-1}) \right|$$

$$\leq \left| -y \tau' \left( \frac{y}{\langle \phi \rangle} \right) \langle \phi \rangle^{-2} \right|. $$

Also

$$\varphi_y(x, y) = \frac{\partial}{\partial y} \tau \left( \frac{y}{\langle \phi \rangle} \right)$$

$$= \tau' \left( \frac{y}{\langle \phi \rangle} \right) \cdot \frac{\partial}{\partial y} (y \langle \phi \rangle^{-1})$$

$$= \tau' \left( \frac{y}{\langle \phi \rangle} \right) \langle \phi \rangle^{-1}. $$

Therefore, since \( \tau' \) is bounded on \( \mathbb{R} \), we can set \( K := 3 \sup_{s \in \mathbb{R}} |\tau'(s)| \) to obtain,

$$\left\| (\varphi_x + i\varphi_y)(x, y) \right\| \leq \frac{K}{3} \left[ \frac{|y|}{\langle \phi \rangle^2} + \frac{1}{\langle \phi \rangle} \right]$$

$$\leq \frac{K}{3} \left[ \frac{2 \langle \phi \rangle^2 + 1}{\langle \phi \rangle^2} \right] \quad \text{(using (6))}$$

$$\leq \frac{K}{\langle \phi \rangle}. $$

\[ \square \]

**Theorem 1.4** Let \( n > \alpha \geq 0, \ f \in \mathfrak{A} \) and \( H \) be of \( (\alpha, \alpha + 1) \)-type \( \mathbb{R} \). Then the integral

$$f(H) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial z} (z - H)^{-1} dx dy$$

is norm convergent and defines an operator in \( \mathfrak{B}(\mathcal{X}) \) with

$$\|f(H)\| \leq c_\alpha \|f\|_{n+1} \quad \text{for some } c_\alpha > 0. \quad (8)$$

\textbf{Proof.} Suppose \( \|(z - H)^{-1}\| \leq c_\alpha \frac{\langle \phi \rangle^\alpha}{\|z\|^{n+1}} \) for all \( z \notin \mathbb{R} \) (hypothesis). We will use the notation \( (x, y) = x + iy := z \).

We observe that by (5), \( \frac{\partial \tilde{f}}{\partial z} \) is continuous and hence the integrand is norm continuous for \( z \notin \mathbb{R} \).
Further,

(A) \( \text{supp}(\varphi) \subseteq \{(x, y) : \tau \left( \frac{y}{\langle \varphi \rangle} \right) > 0\} \)

\( \subseteq \left\{ (x, y) : \frac{|y|}{\langle \varphi \rangle} \leq 2 \right\} \)

\( = \left\{ (x, y) : 0 \leq |y| \leq 2 \langle \varphi \rangle \right\} \)

\( =: V \)

(B) \( \text{supp}(\varphi_x + i\varphi_y) \subseteq \left\{ (x, y) : 1 \leq \frac{|y|}{\langle \varphi \rangle} \leq 2 \right\} \)

\( = \left\{ (x, y) : \langle \varphi \rangle \leq |y| \leq 2 \langle \varphi \rangle \right\} \)

\( =: U \)

(C) For \( z \in \text{supp}(\varphi) \cup \text{supp}(\varphi_x + \varphi_y) \setminus \mathbb{R} \),

\[ \| (z - H)^{-1} \| \leq c \frac{\langle \varphi \rangle^\alpha}{|z|^{\alpha + 1}} \]

\[ \leq c \frac{(1 + |x|^2 + 4 \langle \varphi \rangle^2)^{\alpha/2}}{|y|^{\alpha + 1}} \]

\[ \leq c5^{\alpha/2} \frac{\langle \varphi \rangle^\alpha}{|y|^{\alpha + 1}}. \]

(D) \( \| \varphi_x + i\varphi_y \| (x, y) \leq \frac{K}{\langle \varphi \rangle} \) for some \( K > 0 \), Lemma 1.3.

Also, since \( \varphi \) is bounded, let \( M := \sup_{z \in \mathbb{C}} \{ |\varphi(z)| \} \).

Therefore, using the expansion (5) and the estimates above, we have

\[ \| f(H) \| \leq \frac{c5^{\alpha/2}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{r=0}^{n} |f^{(r)}(x)| |y|^{r} K \langle \varphi \rangle^\alpha \chi_U(z) + M |f^{(n+1)}(x)| |y|^n \chi_V(z) \right) \frac{\langle \varphi \rangle^\alpha}{|y|^{\alpha + 1}} dx dy \]

\[ = \frac{c5^{\alpha/2}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \sum_{r=0}^{n} |f^{(r)}(x)| |y|^{r-\alpha-1} K \langle \varphi \rangle^\alpha \chi_U(z) + M |f^{(n+1)}(x)| |y|^{n-\alpha-1} \langle \varphi \rangle^\alpha \chi_V(z) \right) dx dy. \]

Integrating with respect to \( y \) yields the bound

\[ \| f(H) \| \leq \frac{c\alpha}{\pi} \int_{-\infty}^{\infty} \left( \sum_{r=0}^{n} |f^{(r)}(x)| |\langle \varphi \rangle|^{r-1} + |f^{(n+1)}(x)| \langle \varphi \rangle^n \right) dx \]

\[ = c_{\alpha} \| f \|_{n+1} \quad \text{with} \quad c_{\alpha} := \frac{c5^{\alpha/2}2^{n-\alpha}}{\pi} \cdot \max\{K, M\}. \]
Similar integrals to that in (4) play a central role in the theory of uniform algebras, Gamelin [4].

It may seem from the computation above that our definition of $f(H)$ depends implicitly on the cut-off function $\varphi$ and $n$. However we will prove shortly that $f(H)$ is independent of both $\varphi$ and $n$, provided $n > \alpha$.

**Lemma 1.5** If $F \in C^\infty_c(C)$ and $F(z) = O(|y^\beta|$ as $y \to 0$ for some $\beta > \alpha + 1$, then

$$-\frac{1}{\pi} \int_C \frac{\partial F}{\partial \bar{z}} (z - H)^{-1} dxdy = 0. \quad (9)$$

**Proof.** Let $F$ have support in \{\(z = (x, y) : |x| < N \text{ and } |y| < N\}\} and define $\Omega_\delta$ for small $\delta > 0$ to be the region \{\(z = (x, y) : |x| < N \text{ and } \delta < |y| < N\}\} (see figure 1).

Figure 1: Close up on the support of $F$ by compact regions.
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\[ A := -\frac{1}{\pi} \int_{C} \frac{\partial F}{\partial z} (z - H)^{-1} dxdy \]

\[ = -\lim_{\delta \to 0} \frac{1}{\pi} \int_{\partial F} \frac{\partial F}{\partial \bar{z}} (z - H)^{-1} dxdy \]

\[ = \lim_{\delta \to 0} \frac{i}{2\pi} \int_{\partial \Omega} F(z)(z - H)^{-1} dz \quad \text{(Green's Theorem)} \]

\[ = \lim_{\delta \to 0} \frac{i}{2\pi} \sum_{r=1}^{8} \int_{L_r} F(z)(z - H)^{-1} dz \]

since \([\text{supp } F] \cap [\bigcup_{r=2}^{4} L_r \cup \bigcup_{r=6}^{8} L_r] = \emptyset\).

Now for \((x, y) \in L_1 \cup L_5 \subseteq \mathbb{C} \setminus \mathbb{R}\),

\[ \| (z - H)^{-1} \| \leq c \frac{\langle \phi \rangle^{\alpha}}{\langle z \rangle^{\alpha + 1}} = c \frac{(1 + |\phi|^2 + \delta^2)^{\alpha/2}}{\delta^{\alpha+1}} \leq c \frac{2^{\alpha/2} \langle N \rangle^{\alpha}}{\delta^{\alpha+1}}. \]

Therefore

\[ \| A \| \leq c 2^{\alpha/2} \langle N \rangle^{\alpha} \lim_{\delta \to 0} \int_{-N}^{N} \| F(x + i\delta) + F(x - i\delta) \|^{\alpha-1} dx = 0, \]

since by hypothesis the integrand is \(O(\delta^{\beta-\alpha-1})\).

**Theorem 1.6** The operator \(f(H)\) is independent of \(n\) and the cut-off function \(\varphi\) defined in (3), provided \(n > \alpha\).

**Proof.** \(C_c^\infty(\mathbb{R})\) is dense in \(A\) with respect to each norm \(\| \cdot \|_{n+1}\) [9, Lemma 1.5]. This result together with (8) imply that it is sufficient to prove this for \(f \in C_c^\infty\).

If \(f \in C_c^\infty(\mathbb{R})\) while \(\varphi_1\) and \(\varphi_2\) are cut-off functions define in terms of say \(\tau_1\) and \(\tau_2\), let

\[ \Omega_1 := \left\{ (x, y) : \frac{|y|}{\langle x \rangle} < \epsilon_1 \right\} \quad \text{for some } \epsilon_1 > 0 \]

\[ = \left\{ (x, y) : -\epsilon_1 \langle x \rangle < y < \epsilon_1 \langle x \rangle \right\} \]

\[ \subseteq \left\{ z : \varphi_1(z) = 1 \right\}. \]

Similarly let

\[ \Omega_2 := \left\{ (x, y) : -\epsilon_2 \langle x \rangle < y < \epsilon_2 \langle x \rangle \right\} \quad \text{for some } \epsilon_2 > 0 \]

\[ \subseteq \left\{ z : \varphi_2(z) = 1 \right\}. \]

Now set \(\Omega := \Omega_1 \cap \Omega_2\)

\[ = \left\{ (x, y) : -\epsilon \langle x \rangle < y < \epsilon \langle x \rangle \right\} \quad \text{with } \epsilon := \min\{\epsilon_1, \epsilon_2\}

\[ \neq \emptyset. \]
Then for \( z \in \Omega \),
\[
\tilde{f}_{\varphi_1,n}(z) - \tilde{f}_{\varphi_2,n}(z) = \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} [\varphi_1(z) - \varphi_2(z)]
= 0 \quad \text{since } \varphi_1(z) = \varphi_2(z) = 1 \quad \text{for all } z \in \Omega.
\]

This exceeds the hypothesis of lemma 1.5, so invoking lemma 1.5, we have \( \tilde{f}_{\varphi_1,n}(H) = \tilde{f}_{\varphi_2,n}(H) \). That is \( \tilde{f}_{\varphi,n}(H) \) is independent of \( \varphi \).

On the other hand, if \( m > n > \alpha \) then
\[
\tilde{f}_{\varphi_1,m}(z) - \tilde{f}_{\varphi_1,n}(z) = \left( \sum_{r=n+1}^{m} \frac{f^{(r)}(x)(iy)^r}{r!} \right) [\varphi_1(z)]
= \sum_{r=n+1}^{m} \frac{f^{(r)}(x)(iy)^r}{r!} \varphi_1(z)
=: y^{n+1}K(z) \quad \text{(some bounded } K : \mathbb{C} \to \mathbb{C})
\]
and since \( n + 1 > \alpha + 1 \) we invoke Lemma 1.5 to conclude that \( \tilde{f}_{\varphi_1,m}(H) = \tilde{f}_{\varphi_1,n}(H) \). That is \( \tilde{f}_{\varphi,n}(H) \) is independent of \( n \).

Henceforth we will assume that the condition of theorem 1.6 holds and write \( \tilde{f} \) instead of \( \tilde{f}_{\varphi,n} \) unless a specific cut-off function or \( n \) is needed for some purpose which will be stated.

2 The homomorphism \( \mathfrak{A} \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{X}) \)

In this paper, support of \( f \) will be understood to be the set
\[
\text{supp} (f) := \left\{ x \in \mathbb{R} : f(x) \neq 0 \right\}.
\]
Thus \( \text{supp} (\tilde{f}) \) is a closed set.

**Theorem 2.1** Let \( H \) be an operator of \((\alpha, \alpha + 1) - \text{type } \mathbb{R}\) for some \( \alpha \geq 0 \). If \( f \in C_c^{\infty}(\mathbb{R}) \) has support disjoint from \( \sigma(H) \), then \( f(H) = 0 \).

**Proof.** By regularity of \( \mathfrak{C} \), we can find an open set \( G \) with \( \text{supp} (\tilde{f}) \subset G \) and \( G \cap \sigma(H) = \emptyset \). Since by hypothesis, \( \text{supp} (\tilde{f}) \) is compact, there exists a finite set of smooth curves, \( \{\mathcal{V}_r\}_{r=1}^{m} \) ‘enclosing’ \( \text{supp} (\tilde{f}) \) in \( G \). Thus if we put
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\[ \Gamma := \bigcup_{r=1}^{m} \Upsilon_r \text{, and } D := \text{ins} \Gamma \text{ then} \]

\[
f(H) = -\frac{1}{\pi} \int_{C} \frac{\partial \tilde{f}(z)}{\partial z} (z - H)^{-1} dz
\]

\[
= -\frac{1}{\pi} \int_{D} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} (z - H)^{-1} dz
\]

\[
= \frac{i}{2\pi} \int_{\Gamma} \tilde{f}(z)(z - H)^{-1} dz \quad \text{(Green’s Theorem)}
\]

\[
= \frac{i}{2\pi} \sum_{r=1}^{m} \int_{\Upsilon_r} \tilde{f}(z)(z - H)^{-1} dz
\]

\[
= 0 \text{ since } \tilde{f}(z) = 0 \text{ for all } z \in \Upsilon_r \, r = 1, 2, \ldots, m
\]

\[\square\]

**Corollary 2.2** Let \(H\) be an operator of \((\alpha, \alpha + 1) - \text{type} \, \mathbb{R}\) for some \(\alpha \geq 0\). If \(f \in \mathfrak{A}\) has support disjoint from \(\sigma(H)\) then \(f(H) = 0\).

**Proof.** Follows from theorem 2.1, inequality (8) and density of \(C_{c}^{\infty}(\mathbb{R})\) in \(\mathfrak{A}\) [9, lemma 1.5]. \[\square\]

**Theorem 2.3** If \(f, g \in \mathfrak{A}\) and \(H\) is of \((\alpha, \alpha + 1) - \text{type} \, \mathbb{R}\) then

\[ (fg)(H) = f(H)g(H). \]

**Proof.** We first assume that \(f\) and \(g\) lie in \(C_{c}^{\infty}(\mathbb{R})\). Let \(K := \text{supp} \left( \tilde{f} \right)\) and \(L := \text{supp} \left( \tilde{g} \right)\) so that \(K\) and \(L\) are compact subsets of \(\mathbb{C}\) and write

\[
f(H) = -\frac{1}{\pi} \int_{C} \frac{\partial \tilde{f}(z)}{\partial z} (z - H)^{-1} dx dy, \quad z =: x + iy
\]

and \(g(H) = -\frac{1}{\pi} \int_{C} \frac{\partial \tilde{f}(w)}{\partial w} (w - H)^{-1} du dv, \quad w =: u + iv\)

Then \(f(H)g(H) = \frac{1}{\pi^2} \int \int_{K \times L} \frac{\partial \tilde{f}}{\partial z} \frac{\partial \tilde{g}}{\partial w} (z - H)^{-1} (w - H)^{-1} dxdydv.\)

Using resolvent equation,

\[
(z - H)^{-1}(w - H)^{-1} = (z - w)^{-1}(w - H)^{-1} - (z - w)^{-1}(z - H)^{-1},
\]

we may expand \(f(H)g(H)\) as

\[1_{K \times L} := \{(k, l) : k \in K, l \in L\} \]

\[\square\]
\[
f(H)g(H) = \frac{1}{\pi} \int_{K \times L} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{\partial \tilde{g}}{\partial \bar{w}} [(z - w)^{-1}(w - H)^{-1} - (z - w)^{-1}(z - H)^{-1}] dxdy dudv
\]
\[
= -\frac{1}{\pi} \int_{K \times L} \left( \frac{\partial \tilde{g}}{\partial \bar{w}} (w - H)^{-1} - \frac{1}{\pi} \int_{K} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - w)^{-1} dxdy \right) dudv
\]
\[
-\frac{1}{\pi} \int_{K \times L} \left( \frac{\partial \tilde{f}}{\partial \bar{z}} (z - H)^{-1} - \frac{1}{\pi} \int_{L} \frac{\partial \tilde{g}}{\partial \bar{w}} (z - w)^{-1} dudv \right) dx dy
\]

But
\[
-\frac{1}{\pi} \int_{K} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - w)^{-1} dx dy = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - w)^{-1} dx dy
\]
\[
= \tilde{f}(w) \quad \text{(Cauchy-Green Theorem)}.
\]

Also
\[
\frac{1}{\pi} \int_{L} \frac{\partial \tilde{g}}{\partial \bar{w}} (z - w)^{-1} dudv = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{g}}{\partial \bar{w}} (w - z)^{-1} dudv
\]
\[
= \tilde{g}(z) \quad \text{(Cauchy-Green Theorem)}.
\]

These lead to the identity
\[
f(H)g(H) = -\frac{1}{\pi} \int_{K \times L} \frac{\partial \tilde{g}(w)}{\partial \bar{w}} (w - H)^{-1} \tilde{f}(w) dudv + \frac{1}{\pi} \int_{K \times L} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} \tilde{g}(z)(z - H)^{-1} dx dy
\]
\[
= -\frac{1}{\pi} \int_{K \times L} \frac{\partial \tilde{g}(z)}{\partial \bar{z}} (z - H)^{-1} \tilde{f}(z) dx dy + \frac{1}{\pi} \int_{K \times L} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} \tilde{g}(z)(z - H)^{-1} dx dy
\]
\[
= -\frac{1}{\pi} \int_{K \times L} \left\{ \tilde{f}(z) \frac{\partial \tilde{g}(z)}{\partial \bar{z}} + \frac{\partial \tilde{f}(z)}{\partial \bar{z}} \tilde{g}(z) \right\} (z - H)^{-1} dx dy
\]
\[
= -\frac{1}{\pi} \int_{K \times L} \frac{\partial (\tilde{f} \tilde{g})(z)}{\partial \bar{z}} (z - H)^{-1} dx dy.
\]

In order to prove that
\[
f g(H) = -\frac{1}{\pi} \int_{K \times L} \frac{\partial (\tilde{f} \tilde{g})(z)}{\partial \bar{z}} (z - H)^{-1} dx dy,
\]
we must prove that
\[
-\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial k(z)}{\partial \bar{z}} (z - H)^{-1} dx dy = 0,
\]
where \( k(z) := \tilde{f}(z) \tilde{g}(z) - (\tilde{f} \tilde{g})(z) \). Since \( k \) is of compact support and by Theorem 1.6 and Lemma 1.2 may be assumed to satisfy the hypothesis of Lemma 1.5, this follows by invoking Lemma 1.5.
Finally, suppose that \( f, g \in \mathfrak{A} \) and let \( \phi \in C_c^\infty \) such that
\[
\phi(s) = \begin{cases} 
1, & |s| < 1 \\
0, & |s| > 2
\end{cases}
\]

Set \( \phi_m(s) := \phi(s/m) \) and \( f_m := \phi_m f, g_m := \phi_m g \), and \( h_m := \phi_m^2 f g \).

Then \( f_m \to f, g_m \to g \) and \( h_m \to fg \) in the norm \( ||\cdot||_p \) for some \( p > \alpha \). See proof of [9, Lemma 1.5].

From above we have
\[
h_m(H) = f_m(H)g_m(H) \quad \text{for all } m.
\]

We finally use the density of \( C_c^\infty(\mathbb{R}) \) in \( \mathfrak{A} \) and (8) to complete the proof. \( \square \)

**Lemma 2.4** Let \( g \in \mathfrak{A} \) with \( g = 0 \) on \([0, \infty)\). If \( H \) is of \((\alpha, \alpha + 1)\) type \( \mathbb{R} \) and \( \sigma(H) \subseteq [0, \infty) \) then \( g(H) = 0 \).

**Proof.** For \( \epsilon \in (0, \infty) \), let \( H_\epsilon := \epsilon + H \). Then \( H_\epsilon \) is of \((\alpha, \alpha + 1)\) type \( \mathbb{R} \) (since \((\alpha, \beta)\) type \( \mathbb{R} \) operators are stable under perturbations by reals, [10, Theorem 3.6]). But \( \sigma(H) \subseteq [0, \infty) \) implies that \( \sigma(H_\epsilon) \subseteq [\epsilon, \infty) \), and since \( \text{supp}(g) \subseteq (\infty, 0] \) we must have \( g(H_\epsilon) = 0 \) by Theorem 2.1.

Now
\[
0 = g(H_\epsilon) = -\frac{1}{\pi} \int_C \frac{\partial}{\partial w} \tilde{g}(w)(w - (\epsilon + H))^{-1} dudv
\]
\[
= -\frac{1}{\pi} \int_C \frac{\partial}{\partial z} \tilde{g} (z + \epsilon)(z - H)^{-1} dxdy
\]
\[
= -\frac{1}{\pi} \int_C \frac{\partial}{\partial z} \tilde{g}_\epsilon(z)(z - H)^{-1} dxdy
\]
\[
= g_\epsilon(H)
\]

where \( z := w - \epsilon \) and \( g_\epsilon := \tau_\epsilon g \in \mathfrak{A} \) (since \( \mathfrak{A} \) is invariant under translations [9, Lemma 1.2]).

So by (8)
\[
\|g_\epsilon(H) - g(H)\| = \|g(H)\| \leq c_\alpha \|g_\epsilon - g\|_{n+1}, \quad \text{for some } n > \alpha, \ c_\alpha > 0 \ \text{for all } \epsilon > 0.
\]

Suppose \( g_\epsilon \in \mathcal{G}^{\beta_\epsilon} \) for some \( \beta_\epsilon < 0 \) and \( \epsilon \geq 0 \), where we set \( g_0 := g \). Then
\[
\|g_\epsilon(x)\| \leq c_{r, \epsilon} \langle \varphi \rangle^{\beta_\epsilon - r} \text{ for each } x \in \mathbb{R}.
\]

Let \( \beta := \sup\{\beta_\epsilon : \epsilon \in (0, \infty)\} < 0 \) and \( c := \max_{0 < r \leq n} \sup_{\epsilon \in (0, \infty)} \{c_{r, \epsilon}\} > 0 \).

\( \beta \) and \( c \) exist and are finite, see the proof of [9, Lemma 1.2]. Thus
\[
\|g_\epsilon^{(r)}(x)\| \langle \varphi \rangle^{r-1} \leq c \langle \varphi \rangle^{\beta - r} \langle \varphi \rangle^{r-1} = c \langle \varphi \rangle^{\beta - 1}
\]
and the function $h(x) = c \langle \xi \rangle^{\beta - 1}$ is integrable and
\[ \left| \frac{d^r}{dx^r} (g_\epsilon(x) - g(x)) \right| \langle \xi \rangle^{r-1} \leq h(x) \]
for each $\epsilon$. Therefore by dominated convergence theorem we have
\[ \int_0^\infty \left| g^{(r)}(x) - g^{(r)}(x) \right| \langle \xi \rangle^{r-1} dx \to 0 \]
as $\epsilon \to 0$.

that is $\|g_\epsilon - g\|_{n+1} \to 0$ as $\epsilon \to 0$. Thus $\|g(H)\| = 0$. \hfill \Box

Let $E : C^\infty([0, \infty)) \to C^\infty(\mathbb{R})$ be the Seeley’s extension operator [11]. For $f \in C^\infty([0, \infty))$ such that $Ef \in \mathfrak{A}$ we define $f(H)$ to be $Ef(H)$ where $Ef(H)$ is given by (7) with appropriate condition on $\|(z - H)^{-1}\|$.

**Theorem 2.5** If $f : [0, \infty) \to \mathbb{C}$ is such that
\[ \left| \frac{d^r}{dx^r} f(x) \right| \leq c_r \langle \xi \rangle^{\beta - r} \]
for some $\beta < 0$, for all $r \geq 0$ and for all $x \geq 0$; and $H$ is of $(\alpha, \alpha + 1)$-type $\mathbb{R}$ with $\sigma(H) \subseteq [0, \infty)$, then $f(H)$ is uniquely determined and
\[ \|f(H)\| \leq k \|f\|^+_{n+1}, \quad k > 0, \quad \text{whenever } n > \alpha. \]

**Proof.** Observe that $Ef \in \mathfrak{A}$, where $E$ is Seeley’s extension operator (Lemma 3.3[9]). So that
\[ f(H) \equiv Ef(H) \]
is defined and $f(H) \in \mathfrak{B}(\mathfrak{A})$.

Moreover,
\[
\|f(H)\| = \|(Ef)(H)\| \\
\leq c_{n+1} \|Ef\|_{n+1} \quad \text{[8]} \\
\leq c' c_{n+1} \|f\|^+_{n+1} \quad \text{[9, Theorem 3.4]} \\
=: K \|f\|^+_{n+1}.
\]

Finally if $g \in \mathfrak{A}$ is another extension of $f$, set
\[ h : = g - Ef \in \mathfrak{A} \]
which implies $h = 0$ on $[0, \infty)$ and thus by Lemma 2.4
\[ h(H) = 0. \]

\hfill \Box

**Corollary 2.6** Let $f, g \in C^\infty([0, \infty))$ satisfy (10) with $H$ of $(\alpha, \alpha + 1)$-type $\mathbb{R}$ and $\sigma(H) \subseteq [0, \infty)$. Then
\[ (fg)(H) = f(H)g(H). \]
Proof.

$$(Ef)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t) f(b_k t), & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

and

$$(Eg)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t) g(b_k t), & t < 0 \\ g(t), & t \geq 0 \end{cases}$$

Thus $(Eg)(t)(Ef)(t) := \begin{cases} (\sum_{k=0}^{\infty} a_k \phi(b_k t) g(b_k t)) \left( \sum_{k=0}^{\infty} a_k \phi(b_k t) f(b_k t) \right), & t < 0 \\ g(t)f(t), & t \geq 0 \end{cases}$. Clearly $gf$ satisfies (10) and

$$E(gf)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k \phi(b_k t)(gf)(b_k t), & t < 0 \\ g(t)f(t), & t \geq 0 \end{cases}$$

Thus, $(Eg)(t)(Ef)(t) = E(gf)(t), t \geq 0$.

Therefore by Lemma 2.4 $(Eg)(H)(Ef)(H) = E(gf)(H)$, i.e. $g(H)f(H) = gf(H)$. 

Remark 2.7 Theorem 2.3 and Corollary 2.6 show that the map

$$\kappa: \mathfrak{A} \rightarrow \mathfrak{B}(X)$$

$$f \mapsto f(H)$$

is an algebra homomorphism. We prove one more result to verify that $\kappa$ is a functional calculus.

Theorem 2.8 Let $H$ be an operator of $(\alpha, \alpha + 1)$-type $\mathbb{R}$ for some $\alpha \geq 0$. If $w \notin \mathbb{R}$ and $r_w(x) := (w - x)^{-1}$ for all $x \in \mathbb{R}$ then $r_w \in \mathfrak{A}$ and

$$r_w(H) = (w - H)^{-1}.$$ 

Proof. Clearly $r_w \in \mathfrak{A}$, and without loss of generality, suppose that $\Im w > 0$. For large $m > 0$ define $\Omega_m := \{(x, y) : |x| < m \text{ and } \frac{\Im w}{m} < |y| < 2m\}$.

The boundary of $\Omega_m$ consists of two closed curves, both traversed in the anti-clockwise direction, see figure 2.

With $\tau$ as in definition 1.1, put

$$\varphi(z) := \tau\left(\frac{\lambda |y|}{|y|^2}\right)$$

where $\lambda > 0$ is chosen

1. large enough to ensure that $w \notin \text{supp}(\varphi)$. 


Figure 2: Close up on $C$, over which $r_w(z)$ is integrated.

2. so that for each $m \geq 1$, $|y| < \frac{2|m|}{\lambda} \leq 2m$, for all $(x, y) \in \Omega_m$. Thus, $\langle 1/m \rangle \leq \lambda$. Since $\langle 1/m \rangle < 2$ for all $m \geq 1$ we may assume that $\lambda \geq 2$.

An application of Green’s Theorem yields

$$r_w(H) = -\lim_{m \to \infty} \frac{1}{\pi} \int_{\partial \Omega_m} \frac{\partial \tilde{r}_w(z - H)}{\partial z} - 1 \, dx \, dy$$

$$= \lim_{m \to \infty} \frac{i}{2\pi} \int_{\partial \Omega_m} \tilde{r}_w(z)(z - H)^{-1} \, dz.$$

We next show that

$$\lim_{m \to \infty} \left\| \int_{\partial \Omega_m} \{r_w(z) - \tilde{r}_w(z)\}(z - H)^{-1} \, dz \right\| = 0.$$

$\partial \Omega_m$ consists of four vertical straight lines, two horizontal straight lines and two curves. The integral is estimated separately on each of these.

1. **Vertical lines:** Suppose $\gamma_1$ is the vertical line in the first quadrant. Using Taylor’s approximation theorem to expand $r_w(z)$ at $(m, 0)$, we obtain, for all $z \in \gamma_1$,

$$r_w(z) = \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + R(z; m).$$
with \( R(z; m) := \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}, \quad d = m + \epsilon iy \) for some \( \epsilon \in (0, 1) \).

Therefore, for any \( z \in \gamma_1 \) we have

\[
\begin{align*}
r_w(z) &= \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}.
\end{align*}
\]

which implies

\[
\begin{align*}
|r_w(z) - \tilde{r}_w(z)| &\leq |1 - \varphi(z)r_w(z)| + |\varphi(z)||r_w(z) - \tilde{r}_w(z)| \\
&= c_1 \chi(z) \langle z \rangle^{-1} + |\varphi(z)| \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!} - \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} \\
&\leq c_1 \chi(z) \langle z \rangle^{-1} + c_w \frac{||y||^{n+1}}{\langle y \rangle^{n+2}}
\end{align*}
\]

where \( \chi(z) := \begin{cases} 1 & \text{if } \langle z \rangle < \lambda ||y|| \\ 0 & \text{otherwise.} \end{cases} \)

But \( z = m + iy, \ d = m + i\epsilon y \) implies \( \langle z \rangle \geq \langle m \rangle \) and \( \langle m \rangle \leq \langle \eta \rangle \). So,

\[
|r_w(z) - \tilde{r}_w(z)| \leq c_1 \chi(z) \langle m \rangle^{-1} + c_w \frac{||y||^{n+1}}{\langle m \rangle^{n+2}}
\]

Also, for \( z := m + iy \in \gamma_1, \quad \frac{\langle m \rangle}{m} \leq ||y|| \leq 2m \) and hence

\[
\begin{align*}
\langle y \rangle^2 &= 1 + ||m||^2 + ||y||^2 \\
&\leq 1 + m^2 + 4m^2 \\
&\leq 5 \langle m \rangle^2.
\end{align*}
\]

Therefore

\[
\|(z - H)^{-1}\| \leq \frac{c_1^{5/2} \langle m \rangle^{\alpha^2}}{||y||^{\alpha+1}} \quad \text{for some } c > 0.
\]
Hence

\[
\left\| \int_{\gamma_1} \{ r_w(z) - \tilde{r}_w(z) \} (z - H)^{-1} dz \right\|
\]

\[
\leq cc_1 5^{\alpha/2} \int_{\frac{c}{y}}^{2m} \frac{\langle m \rangle}{y^{\alpha+1}} dy + cc_2 5^{\alpha/2} \int_{\frac{c}{y}}^{2m} \frac{\langle m \rangle}{y^{\alpha-n-2}} dy
\]

\[
\leq cc_1 5^{\alpha/2} \frac{\langle m \rangle}{\langle m \rangle^{\alpha-1}} \left( \frac{\lambda_{\alpha+1}}{\langle m \rangle^{\alpha+1}} \right) 2m - \frac{\langle m \rangle}{\lambda} + cc_2 5^{\alpha/2} \frac{\langle m \rangle^{\alpha-2}}{\langle m \rangle^{\alpha-n-2}} \frac{\langle n \rangle^{\alpha-2}}{\langle n \rangle^{\alpha-n-2}} 2m - \frac{\langle m \rangle}{m}
\]

\[
= (m^{-1}) \left\{ c_1 (1/m)^{-2} \left| \langle m \rangle \right| - c_2 (1/m)^{\alpha-n-2} \left| \langle m \rangle \right| - \frac{\langle m \rangle}{m} \right\}
\]

\[
= O(m^{-1}) \quad \text{as} \quad m \to \infty, \quad \text{provided} \quad n > \alpha.
\]

The estimate is valid for all vertical lines.

2. The curves: Let \( \gamma_2 \) be the curve in the upper half plane, i.e. \( \gamma_2 := \{(x, y) : y = \frac{\langle \omega \rangle}{m}\} \).

Since \( \frac{1}{m} \langle \omega \rangle < \frac{1}{\lambda} \langle \omega \rangle \) for all \( m > \lambda \), \( \varphi(z) = 1 \) for all \( z \in \gamma_2 \) and \( m > \lambda \).

Therefore using Taylor’s approximation at \( (x, 0) \) with \( d := x + \epsilon iy \) for some \( \epsilon \in (0, 1) \) we have,

\[
\left\| r_w(z) - \tilde{r}_w(z) \right\| \leq (1 - \varphi(z)) r_w(z) + \varphi(z) \left| r_w(z) - \tilde{r}_w(z) \right| \frac{\varphi(z)}{\varphi(z)}
\]

\[
= \varphi(z) \left| r_w(z) - \tilde{r}_w(z) \right| \frac{\varphi(z)}{\varphi(z)}
\]

\[
\leq c_w \frac{r_w^{(n+1)}(d)}{\langle \omega \rangle^{n+1}} \frac{\langle n+1 \rangle}{(n+2)!}
\]

\[
\leq c_w \frac{\langle m \rangle^{n+1}}{\langle \omega \rangle^{n+2}}, \quad z \in \gamma_2, \quad m > \lambda
\]

where \( \langle \omega \rangle \geq \langle \omega \rangle \). Also, for \( z \in \gamma_2 \),

\[
\langle \omega \rangle^2 = 1 + |q|^2 + |y|^2
\]

\[
= \langle \omega \rangle^2 + \frac{\langle \omega \rangle^2}{m^2}
\]

\[
= \frac{\langle m \rangle^2}{m^2} \langle \omega \rangle^2.
\]
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Hence \(\| (z - H)^{-1} \| \leq \frac{c \delta^\alpha \phi^\alpha}{m^n |y|^{n+1}}\) for some \(c > 0\) and

\[
\left\| \int_{\gamma_2} \{ r_w(z) - \tilde{r}_w(z)\} (z - H)^{-1} dz \right\| \leq c_\omega \int_{\gamma_2} \frac{|y|^{n+1}}{\langle \phi \rangle |z|^{n+2}} \frac{\langle m \rangle^\alpha \langle \phi \rangle^\alpha}{m^n |y|^{n+1}} dz
\]

\[
= c_1 \int_{\gamma_2} \frac{\langle m \rangle^\alpha}{m^n} \langle \phi \rangle^\alpha \langle x \rangle^{n-2} \langle |y|^{n-\alpha} dz
\]

\[
= c_1 \int_{\gamma_2} \frac{\langle m \rangle^\alpha}{m^n} \langle \phi \rangle^\alpha \langle x \rangle^{n-2} \langle |y|^{n-\alpha} dz
\]

\[
= c_1 \int_{\gamma_2} \frac{\langle m \rangle^\alpha}{m^n} \langle \phi \rangle^{-2} dz
\]

\[
= O(m^{\alpha-n}) \text{ as } m \to \infty
\]

provided \(n > \alpha\). The estimate here is also valid for the other curve.

3. The horizontal lines
Let \(\gamma_3\) be the horizontal line in the upper half plane, i.e. \(\gamma_3 := \{(x, y) : y = 2m\}\). Now \(\text{supp} (\varphi) \subset \left\{ (x, y) : \frac{x}{y} \leq 2 \right\}\). Thus \(\varphi(z) = 0\) for all \(z \in \Omega_m\) with \(|y| > \frac{2 \langle m \rangle^2}{\lambda}\).

So, for \(z \in \gamma_3, \varphi(z) = 0\) if \(2m > \frac{2 \langle m \rangle^2}{\lambda}\), that is \(\lambda > \langle 1/m \rangle\).

Therefore if we choose \(m\) large enough so that \(\lambda > \langle 1/m \rangle\), then for \(z \in \gamma_3,

\[
|r_w(z) - \tilde{r}_w(z)| = |r_w(z)|
\]

\[
\leq \frac{c_1}{\langle \phi \rangle}
\]

\[
= \frac{c_1}{\sqrt{1 + |x|^2 + |y|^2}}
\]

\[
\leq \frac{c_1}{\langle m \rangle^2}.
\]

Also, for \(z \in \gamma_3,

\[
\langle \phi \rangle^2 = 1 + |x|^2 + |y|^2
\]

\[
\leq 1 + m^2 + 4m^2
\]

\[
\leq 5 \langle m \rangle^2.
\]
Hence \(|(z - H)^{-1}| \leq \frac{c_5^{\alpha/2} |m|^\alpha}{2^m(m+1)}\) for some \(c > 0\) and
\[
\left\| \int_{\gamma_3} \{r_w(z) - \tilde{r}_w(z)\} (z - H)^{-1} dz \right\| \leq c_1 c_5^{\alpha/2} \int_{\gamma_3} \frac{(\partial)^{\alpha}}{2m} (2m)^{\alpha+1} dz \\
\leq c_2 m^{-2} (1/2m)^{-1} (1/m)^{\alpha} \int_{-m}^{m} dx \\
= 2c_2 m^{-2} (1/2m)^{-1} (1/m)^{\alpha} m \\
= O(m^{-1}) \text{ as } m \to \infty
\]
provided \(n > \alpha\). The estimate here is also valid for the other horizontal line.

Combining all the cases we obtain
\[
r_w(H) = \frac{i}{2\pi} \lim_{m \to \infty} \int_{\partial \Omega_m} r_w(z)(z - H)^{-1} dz.
\]
The integrand is holomorphic on and inside the part of \(\partial \Omega_m\) in the lower half plane, so the contribution of that integral is zero by Cauchy’s theorem. The integrand is meromorphic in the upper half plane with a single pole at \(z = w\).

Therefore
\[
r_w(H) = -\text{Res}_{z=w}\{r_w(z)(z - H)^{-1}\} \\
= (w - H)^{-1}
\]
where \(\text{Res}_{z=w} f(z)\) denotes the residue of \(f\) at the pole \(w\).

**Definition 2.9** By a \(\mathfrak{A}\)-functional calculus for an operator \(H\) of \((\alpha, \alpha + 1)\)-type \(\mathbb{R}\) we will mean a continuous linear map \(\kappa\) from \(\mathfrak{A}\) into \(\mathfrak{B}(\mathcal{X})\) such that

1. \(\kappa(fg) = \kappa(f)\kappa(g)\), for all \(f, g, \in \mathfrak{A}\).

2. If \(w \notin \mathbb{R}\) then \(r_w \in \mathfrak{A}\) and \(\kappa(r_w) = (w - H)^{-1}\) (\(r_w\) is defined in Theorem 2.8).

Note that in this definition \(\kappa(f) \equiv f(H)\).

**Lemma 2.10** Let \(f \in C_0(\mathbb{R})\), \(H\) a closed operator with \(\sigma(H) \subset \mathbb{R}\) and \(\lambda \in \mathbb{R}\setminus\{0\}\) such that \(f^{-1}(\lambda) \neq \emptyset\) and \(f^{-1}(\lambda) \cap \sigma(H) = \emptyset\). Then there exists a smooth function \(\phi \in C^\infty(\mathbb{R})\) and a neighbourhood \(G\) of \(\sigma(H)\) such that
\[
\phi(t) = \begin{cases} 
0 & \text{if } t \in f^{-1}(\lambda) \\
1 & \text{if } t \in G.
\end{cases}
\]
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**Proof.** Let \(x_0 \in \mathbb{R}\) be such that \(f(x_0) = \lambda\) and \(d := \text{dist}(x_0, \sigma(H)) > 0\). Choose \(\epsilon_0 \in \mathbb{R}: 0 < \epsilon_0 < d\). Let \(G_0\) be an open set such that
\[
[x_0 - \epsilon_0, x_0 + \epsilon_0] \subset G_0 \subset [x_0 - d, x_0 + d].
\]

We can choose a smooth function \(\psi_0\) such that
\[
\psi_0(t) = \begin{cases} 
1 & \text{if } t \in [x_0 - \epsilon_0, x_0 + \epsilon] \\
0 & \text{if } t \in \mathbb{R} \setminus G_0.
\end{cases}
\]
Next, set \(\phi_0 := 1 - \psi_0\), then clearly \(\phi_0\) is smooth and
\[
\phi_0(t) = \begin{cases} 
0 & \text{if } t = x_0 \\
1 & \text{if } t \in \mathbb{R} \setminus G_0.
\end{cases}
\]

\(G_0 := (a, b)\)

Now set \(O_{x_0} := (x_0 - \epsilon_0, x_0 + \epsilon_0)\). Similarly choose open sets \(O_x\) for each \(x \in f^{-1}(\lambda)\). Since \(\lambda \neq 0\) and \(f(x) \to 0\) as \(|x| \to \infty\), \(f^{-1}(\lambda)\) is a compact set and \(\{O_x : x \in f^{-1}(\lambda)\}\) is an open cover for \(f^{-1}(\lambda)\), hence we can find a finite sub-cover \(\{O_i : i = 1, \ldots, m\} \subset \{O_x : x \in f^{-1}(\lambda)\}\). Corresponding to each \(O_i\), let \(\phi_i\) be the smooth function constructed above. So \(\phi_i = 1\) on \(O_i\), and \(\phi_i = 1\) on \(\mathbb{R} \setminus G_i\). Finally, put \(\phi := \prod_{i=1}^{m} \phi_i, \ G := (\mathbb{R} \cup \bigcup_{i=1}^{m} G_i)^c \supset \sigma(H)\), whence
\[
1. \phi \text{ is smooth on } \mathbb{R}.
2. \phi \equiv 0 \text{ on } f^{-1}(\lambda).
3. \phi \equiv 1 \text{ on } G.
\]

**Theorem 2.11 (Spectral Mapping Theorem)** Let \(f \in \mathfrak{A}\) and \(H\) is of \((\alpha, \alpha + 1)\) type \(R\), then
\[
f(\sigma(H)) = \sigma(f(H)).
\]

**Proof.** Let \(\lambda \in \sigma(H) \subset \mathbb{R}\) and suppose if possible
\[
f(\lambda) \not\in \sigma(f(H)). \tag{12}
\]
Then \([f(\lambda) - f(H)]^{-1} \in \mathfrak{B}(\mathcal{X})\).

If \(\hat{f}_\lambda(x) := \left\{ \begin{array}{ll}
\frac{f(\lambda) - f(x)}{\lambda - x}, & x \neq \lambda \\
f'(\lambda), & x = \lambda,
\end{array} \right.\)
then \(\hat{f}_\lambda \in \mathfrak{A} \tag{9, Theorem 2.6}, \) and
\[
(\lambda - H)\hat{f}_\lambda(H)(i - H)^{-1} = (f(\lambda) - f(H))(i - H)^{-1}.
\]
Thus

\[(\lambda - H) f_\lambda(H)(i - H)^{-1}(i - H)(f(\lambda) - f(H))^{-1} = \]

\[= (f(\lambda) - f(H))(i - H)^{-1}(i - H)(f(\lambda) - f(H))^{-1} \]

\[\iff (\lambda - H) f_\lambda(H)(f(\lambda) - f(H))^{-1} = I.\]

Therefore \((\lambda - H)^{-1} = f_\lambda(H)(f(\lambda) - f(H))^{-1} \in \mathcal{B}(\mathcal{X})\) \(!!^2\)

This contradicts the choice of \(\lambda\). Hence (12) is not possible. Thus \(f(\lambda) \in \sigma(f(H))\) implies \(f(\sigma(H)) \subseteq \sigma(f(H))\).

Conversely, if \(\lambda \notin f(\sigma(H))\) then \(h(x) := \frac{1}{\lambda - f(x)}\) is finite for all \(x \in \sigma(H)\). Moreover at each \(x \in \sigma(H)\) (and \(x \in G\) where \(G\) is the neighbourhood of \(\sigma(H)\) constructed in lemma 2.10)

\[h'(x) = [\lambda - f(x)]^{-2} f'(x)\]

\[= [h(x)]^2 f'(x)\]

\[h^{(2)}(x) = f^{(2)}(x)[h(x)]^2 + 2 f'(x)h(x)[h(x)]^2 f'\]

\[= f^{(2)}[h(x)]^2 + 2[f'(x)]^2[h(x)]^3\]

\[h^{(3)}(x) = f^{(3)}(x)[h(x)]^2 + f^{(2)}(x)2f'(x)h(x)[h(x)]^2 f' +\]

\[+ 2\{2f'(x)f^{(2)}(x)[h(x)]^3 + [f'(x)]^2 + 3[h(x)]^2[f(h)]^2 f'(x)\}\]

\[= f^{(3)}[h(x)]^2 + 6f'(x)f^{(2)}(x)[h(x)]^3 + 6[f'(x)]^3[h(x)]^4\]

\[
\vdots \quad \vdots \\
\]

\[h^{(m)}(x) = \sum_{k=2}^{m+1} [h(x)]^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [f^{(i)}(x)]^{p(s,i)} l_s\]

where \(l_s \in \mathbb{Z}, 1 \leq r(k) < m, 0 \leq p(s,i) \leq m\) and \(\sum_{i=1}^{m} ip(s,i) = m\). Therefore since \(f \in \mathcal{A}\), we can find some \(\beta < 0\) such that

\[|h^{(m)}(x)| \leq \sum_{k=2}^{m+1} [h(x)]^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} |f^{(i)}(x)|^{p(s,i)} ||_s\]

\[\leq \sum_{k=2}^{m+1} [h(x)]^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [c_i \langle \phi \rangle^{\beta-i}]^{p(s,i)} ||_s\]

\[\leq \langle \phi \rangle^{\beta-m} \sum_{k=2}^{m+1} [h(x)]^k b_k\]

\[\leq c \langle \phi \rangle^{\beta-m}, \quad c > 0, \ \beta < 0\] (13)

(Here we have used the fact that \(\sum_{i=1}^{m} p(s,i) \geq 1\) and \(|h|_G < \infty\).)

If \(\phi\) is the smooth function such that

\[
\phi(t) = \begin{cases} 
0 & \text{if } t \in f^{-1}(\lambda) \\
1 & \text{if } t \in G
\end{cases}
\]

\(^2\text{We denote contradiction by }!!\)
also constructed in lemma 2.10, set
\[ g(x) := (i - x)^{-1}\phi(x)h(x). \]

Then using (13) and since \((x + c)(w - x)^{-1}q, (q + c)(w - x)^{-1} \in \mathfrak{A}\) for \(q \in \mathfrak{A}\)
and \(c, w \in \mathbb{C}\) with \(\Re w \neq 0\) [9, Lemma 2.5] we conclude that \(g \in \mathfrak{A}\) and
\[ (\lambda - f(H))g(H)(i - H) = I. \]
That is, \(\lambda - f(H)\) has an inverse. Therefore, \(\lambda \notin \sigma(f(H))\). Hence
\(\sigma(f(H)) \subseteq f(\sigma(H))\). □

3 Extending the functional calculus to \(C_0(\mathbb{R})\)

Let \(C_0(\mathbb{R})\) denote the algebra of all continuous functions \(f : \mathbb{R} \to \mathbb{C}\) such that
\(f(x) \to 0\) as \(|x| \to \infty\) with the supremum norm \(||f||_{\infty}\). Then \(\mathfrak{A}\) is a dense
sub-algebra of \(C_0(\mathbb{R})\) [10, Corollary 2.4].

In this section we extend the \(\mathfrak{A}\)-functional calculus to \(C_0(\mathbb{R})\). For more
general extensions, see DeLaubenfels [1]. First, we have the following preliminaries.

**Lemma 3.1** If \(f \in \mathfrak{A}\) and \(H\) is self-adjoint on Hilbert space \(\mathcal{H}\), then
\(||f(H)|| \leq ||f||_{\infty}\).

**Proof.** First, observe that \(H\) is of \((0, 1) - type \mathbb{R} \). Also from (7),
\[ \bar{f}(H) = f(H)^* \]
in this case. Now choose \(d \in \mathbb{R}\) such that \(d > ||f||_{\infty}\) and set
\[ g(t) := d - \sqrt{(d^2 - |f(t)|^2)} \]
then clearly \(0 \leq g \in \mathfrak{A}\), and
\[ (d - g(t))^2 = d^2 - |f(t)|^2, \text{ for each } t \in \mathbb{R}. \]
so
\[ f\bar{f} - 2dg + g^2 = 0 \in \mathfrak{A}. \]
Thus
\[ f(H)^* f(H) - dg(H) - dg(H)^* + g(H)^* g(H) = 0 \]
implies
\[ f(H)^* f(H) + (d - g(H))^* (d - g(H)) = d^2. \]
If \( \psi \in \mathcal{H} \), then
\[
\|f(H)\psi\|^2 + \|(d - g(H))\psi\|^2 = \|d\psi\|^2 + \|\{d - g(H)\}\psi\|^2 = \|d\psi\|^2
\]
and therefore
\[
\|f(H)\psi\| \leq \|d\psi\|.
\]
\[\square\]

We are now in a position to describe the \( \mathfrak{A} \)-functional calculus for a self-adjoint operator in a standard fashion.

**Corollary 3.2** If \( f \in \mathfrak{A} \) and \( H \) is self-adjoint on Hilbert space \( \mathcal{H} \), then the functional calculus
\[
\kappa : \mathfrak{A} \ni f \mapsto f(H) \in \mathcal{B}(\mathcal{H})
\]
can be extended to a unique map
\[
\tilde{\kappa} : C_0(\mathbb{R}) \ni f \mapsto f(H) \in \mathcal{B}(\mathcal{H})
\]
such that:

1. \( \tilde{\kappa} \) is an algebra homomorphism.
2. \( \tilde{f}(H) = f(H) \).
3. \( \|f(H)\| \leq \|f\|_{\infty} \).
4. if \( w \in \mathbb{C} \setminus \mathbb{R} \) and \( r_w := (w - s)^{-1} \) then \( r_w(H) = (w - H)^{-1} \).

**Proof.** The existence follows from Theorem 2.3, Corollary 2.6, Theorem 2.8 and Lemma 3.1. So we need only to establish the uniqueness.

Suppose \( \eta \) is another extension of \( \kappa \) to \( C_0(\mathbb{R}) \) and let \( \mathfrak{X} \subseteq C_0((R) \) be the set of \( f \) for which \( \tilde{\kappa}(f) = \eta(f) \). Then \( \mathfrak{X} \) is norm closed sub-algebra of \( C_0(\mathbb{R}) \) which contains \( r_w \) for all \( w \not\in \mathbb{R} \). Thus whenever \( x, y \in \mathbb{R} \),
\[
x \neq y \iff r_w(x) \neq r_w(y) \quad \text{for some } w \not\in \mathbb{R}
\]
Therefore, by Stone - Weierstrass Theorem, \( \mathfrak{X} = C_0(\mathbb{R}) \). \( \square \)

**Remark 3.3** \( H \) is of \((0, 1)\)-type \( \mathbb{R} \) with the constant \( c = 1 \) if and only if \( iH \) is a generator of a one-parameter group of isometries on \( \mathfrak{X} \)\[10, \text{Theorem 3.1}\]. This together with Corollary 3.2 provide a proof to a version of the spectral theorem for a self-adjoint operator on a Hilbert space, which asserts:

\( iH \) generates a uniformly bounded strongly continuous group if and only if \( H \) has a \( C_0(\mathbb{R}) \) functional calculus.
A functional calculus for \((\alpha, \alpha + 1) - \text{type } \mathbb{R} \) operators

The most natural infinite dimensional analogue of a diagonalizable matrix is a **scalar operator** (short for spectral operator of scalar type in the sense of Dunford [3, Chapter XVIII]). For an operator \(H\) with real spectrum, this means that there exits a projection-valued measure \(F\) such that

\[
Hx = \int_{\mathbb{R}} tdF(t)x
\]

with maximal domain.

The class of scalar operators includes (but is not limited to) self-adjoint operators on a Hilbert space. However on a general Banach space, it is hard to find a scalar operator. If \(H\) is an operator with \(\sigma(H) \subset \mathbb{R}\) and acting on a reflexive Banach space \(\mathcal{X}\), then \(H\) is scalar if and only if \(iH\) generates a uniformly bounded strongly continuous group [8, page 155]. So, via the spectral theorem, a self-adjoint operator \(H\) on a Hilbert space \(\mathcal{H}\) is scalar if and only if \(H\) has a \(C_0(\mathbb{R})\) functional calculus. In fact this is true in general. That is;

an operator acting on a reflexive Banach space is scalar if and only if it has a \(C_0(\mathbb{R})\) functional calculus [2, Theorem 6.10].

In the light of the forgoing, it is therefore reasonable to have the following conjecture:

**Conjecture 3.4** A densely define closed linear operator \(H\), acting on a reflexive Banach space \(\mathcal{X}\), is scalar if it is of \((0, 1) - \text{type } \mathbb{R}\) and \(\|f(H)\| \leq \|f\|_{\infty}\) for each \(f \in \mathfrak{A}\).

**References**


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