On Weak Separation Axioms Associated with $(1, 2)^\ast$-Sg-Closed Sets

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Abstract

The aim of this paper is to introduce separation axioms using $(1, 2)^\ast$-semi-generalised open sets and $(1, 2)^\ast-\psi$-open sets.

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1 Introduction

Levine [4], Mashhour et al [6] and Njastad [7] have introduced the concepts of semi-open sets, preopen sets and $\alpha$-open sets respectively. Levine [5] introduced generalised closed sets and studied their properties. Bhattacharya and Lahiri [2] introduced semi-generalised closed sets. Thivagar et al [9] have introduced the concepts of $(1, 2)^\ast$-semi-open sets, $(1, 2)^\ast$-generalised-closed sets, $(1, 2)^\ast$-semi-generalised closed sets in bitopological spaces.

In this paper we introduce the generalised forms of $(1, 2)^\ast$-semi-separation axioms using $(1, 2)^\ast$-semi-generalised open sets called $(1, 2)^\ast$-semi-generalised -$T_i$ (briefly denoted by $(1, 2)^\ast$-sg-$T_i$) axioms. Also the concepts of $(1, 2)^\ast$-\psi-open sets are defined in $(1, 2)^\ast$-bitopological spaces and used to define another class of separation axioms called $(1, 2)^\ast$-\psi-separation axioms which are weaker than the class of $(1, 2)^\ast$-semi-$T_i$ axioms and stronger than $(1, 2)^\ast$-sg-$T_i$ axioms, $i = 0, 1, 2$. We also study their basic properties and relative preservation properties of these spaces.
2 Preliminaries

Throughout this paper \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) represent bitopological spaces on which no separation axioms are assumed unless otherwise mentioned.

**Definition 2.1** [9] A subset \(S\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be \(\tau_{1,2}\)-open if \(S = A \cup B\) where \(A \in \tau_1\) and \(B \in \tau_2\). A subset \(S\) of \(X\) is said to be \(\tau_{1,2}\)-closed if the complement of \(S\) is \(\tau_{1,2}\)-open.

**Definition 2.2** [9] Let \(S\) be a subset of \(X\). Then

(i) The \(\tau_{1,2}\)-interior of \(S\), denoted by \(\tau_{1,2}\)-int\((S)\) is defined by \(\cup\{G/G \subset S\text{ and } G \text{ is } \tau_{1,2}\text{-open}\}\).

(ii) The \(\tau_{1,2}\)-closure of \(S\) denoted by \(\tau_{1,2}\)-cl\((S)\) is defined by \(\cap\{F/S \subset F\text{ and } F \text{ is } \tau_{1,2}\text{-closed}\}\).

**Remark 2.3**

(i) \(\tau_{1,2}\)-int\((S)\) is \(\tau_{1,2}\)-open for each \(S \subset X\) and \(\tau_{1,2}\)-cl\((S)\) is \(\tau_{1,2}\)-closed for each \(S \subset X\).

(ii) A set \(S \subset X\) is \(\tau_{1,2}\)-open iff \(S = \tau_{1,2}\)-int\((S)\) and is \(\tau_{1,2}\)-closed iff \(S = \tau_{1,2}\)-cl\((S)\).

(iii) \(\tau_{1,2}\)-int\((S) = \text{int}_{\tau_1}(S) \cup \text{int}_{\tau_2}(S)\) and \(\tau_{1,2}\)-cl\((S) = \text{cl}_{\tau_1}(S) \cap \text{cl}_{\tau_2}(S)\) for any \(S \subset X\)

(iv) For any family \(\{S_i/i \in I\}\) of subsets of \(X\) we have

(a) \(\bigcup_i \tau_{1,2}\)-int\((S_i) \subset \tau_{1,2}\)-int\((\bigcup_i S_i)\)

(b) \(\bigcup_i \tau_{1,2}\)-cl\((S_i) \subset \tau_{1,2}\)-cl\((\bigcup_i S_i)\)

(c) \(\tau_{1,2}\)-int\((\bigcap_i S_i) \subset \bigcap_i \tau_{1,2}\)-int\((S_i)\)

(d) \(\tau_{1,2}\)-cl\((\bigcap_i S_i) \subset \bigcap_i \tau_{1,2}\)-cl\((S_i)\)

(v) \(\tau_{1,2}\)-open sets need not form a topology.

We recall the following definitions which are useful in the sequel.

**Definition 2.4** [8] A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called

(i) \((1,2)^*\)-semi-open if \(A \subseteq \tau_{1,2}\)-cl\((\tau_{1,2}\)-int\((A)\))

(ii) \((1,2)^*\)-preopen if \(A \subseteq \tau_{1,2}\)-int\((\tau_{1,2}\)-cl\((A)\))
(iii) \((1,2)^*\alpha\)-open if \(A \subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)))\)

(iv) \((1,2)^*\)-semi-closed if \(A^c\) is \((1,2)^*\)-semi-open.

(v) \((1,2)^*\)-generalised closed (briefly \((1,2)^*\)-g-closed) if \(\tau_{1,2}\text{-cl}(A) \subset U\) whenever \(A \subset U\) and \(U\) is \(\tau_{1,2}\)-open in \(X\).

**Definition 2.5**  
(i) The \((1,2)^*\)-semi-closure (resp. \((1,2)^*\)-semi-generalised closure) of a subset \(A\) of \(X\), denoted by \((1,2)^*\text{-scl}(A)\) (resp. \((1,2)^*\text{-sgcl}(A)\)), is defined to be the intersection of all \((1,2)^*\)-semi-closed (resp. \((1,2)^*\)-semi-generalised closed) sets containing \(A\).

(ii) The \((1,2)^*\)-semi-interior of a subset \(A\) of \(X\), (resp. \((1,2)^*\)-semi-generalised interior) denoted by \((1,2)^*\text{-sint}(A)\) (resp. \((1,2)^*\text{-sgint}(A)\)), is defined to be the union of all \((1,2)^*\)-semi-open (resp. \((1,2)^*\)-semi-generalised open) sets contained in \(A\).

**Definition 2.6** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called

(i) \((1,2)^*\)-semi-generalised closed (briefly \((1,2)^*\)-sg-closed) if \((1,2)^*\text{-scl}(A) \subset U\) whenever \(A \subset U\) and \(U\) is \((1,2)^*\)-semi-open in \(X\).

(ii) \((1,2)^*\)-sg-open if \(A^c\) is \((1,2)^*\)-sg-closed.

**Remark 2.7**

(i) Since arbitrary union (resp. intersection) of \((1,2)^*\)-semi-open (resp. \((1,2)^*\)-semi-closed ) sets is \((1,2)^*\)-semi-open (resp. \((1,2)^*\)-semi-closed), \((1,2)^*\)-sint\(A\) (resp. \((1,2)^*\text{-scl}(A)\)) is \((1,2)^*\)-semi-open (resp. \((1,2)^*\)-semi-closed).

(ii) For a bitopological space \((X, \tau_1, \tau_2)\), a subset \(A\) of \(X\) is \((1,2)^*\)-semi-open (resp. \((1,2)^*\)-semi-closed) if and only if \((1,2)^*\text{-sint}(A)\) (resp. \((1,2)^*\text{-scl}(A)\)) = \(A\).

Let us recall the following definitions which will be useful in the sequel.

**Definition 2.8** A bitopological space \((X, \tau_1, \tau_2)\) is called a

(i) \((1,2)^*\)-T\(_{1/2}\) space if every \((1,2)^*\)-g-closed set is \(\tau_{1,2}\)-closed.

(ii) \((1,2)^*\)-semi-T\(_{1/2}\) space if every \((1,2)^*\)-sg-closed set is \((1,2)^*\)-semi-closed.

(iii) \((1,2)^*\)-semi-T\(_0\) space if to each pair of distinct points \(x, y\) of \(X\), there exists a \((1,2)^*\)-semi-open set containing one but not the other.

(iv) \((1,2)^*\)-semi-T\(_1\) space if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of \((1,2)^*\)-semi-open sets, one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).
(v) $(1,2)^*\text{-semi-}$-$T_2$ space if to each pair of distinct points $x, y$ of $X$, there exists a pair of disjoint $(1,2)^*\text{-semi-open}$ sets $U$ and $V$ such that $x \in U$ and $y \in V$.

3 \hspace{1cm} (1,2)^*\text{-Semi-Generalised Separation Axioms}

In this section, we define and study some new separation axioms using $(1,2)^*\text{-sg}$-open sets which are weaker than $(1,2)^*\text{-semi-separation}$ axioms.

**Definition 3.1** A bitopological space $(X, \tau_1, \tau_2)$ is called a $(1,2)^*\text{-semi-generalised}$-$T_0$ (briefly written as $(1,2)^*\text{-sg}$-$T_0$) space iff to each pair of distinct points $x, y$ of $X$, there exists a $(1,2)^*\text{-sg}$-open set containing one but not the other.

Clearly, every $(1,2)^*\text{-semi-}$-$T_0$ space is a $(1,2)^*\text{-sg}$-$T_0$ space since every $(1,2)^*\text{-semi}$-open set is a $(1,2)^*\text{-sg}$-open set. The converse is not true in general.

**Example 3.2** Let $X = \{a, b, c\}$; $\tau_1 = \{\phi, \{a, c\}, X\}$ $\tau_2 = \{\phi, \{b, c\}, X\}$; $\tau_{1,2}$-open sets $= \{\phi, \{a, c\}, \{b, c\}, X\} = (1,2)^*\text{-semi-open}$ sets; $(1,2)^*\text{-sg}$-open sets $= \{\phi, \{a, c\}, \{b, c\}, \{c\}, X\}$. $(X, \tau_1, \tau_2)$ is $(1,2)^*\text{-sg}$-$T_0$ but not $(1,2)^*\text{-semi}$-$T_0$.

Now we characterise $(1,2)^*\text{-sg}$-$T_0$ spaces.

**Theorem 3.3** If in any bitopological space, $(X, \tau_1, \tau_2)$, $(1,2)^*\text{-semi}$-generalised closures of distinct points are distinct, then $(X, \tau_1, \tau_2)$ is $(1,2)^*\text{-sg}$-$T_0$.

**Proof** Let $x, y \in X$ and $x \neq y$. By the hypothesis, $(1,2)^*\text{-sgcl}(\{x\}) \neq (1,2)^*\text{-sgcl}(\{y\})$. Then there exists a point $z \in X$ such that $z$ belongs to exactly one of the two sets, say $(1,2)^*\text{-sgcl}(\{y\})$ but not to $(1,2)^*\text{-sgcl}(\{x\})$. If $y \in (1,2)^*\text{-sgcl}(\{x\})$, then $(1,2)^*\text{-sgcl}(\{y\}) \subseteq (1,2)^*\text{-sgcl}(\{x\})$ which implies $y \in (1,2)^*\text{-sgcl}(\{x\})$, a contradiction. So $y \in X - (1,2)^*\text{-sgcl}(\{x\})$, a $(1,2)^*\text{-sg}$-open set which does not contain $x$. This shows that $(X, \tau_1, \tau_2)$ is $(1,2)^*\text{-sg}$-$T_0$.

**Theorem 3.4** In any bitopological space $(X, \tau_1, \tau_2)$, $(1,2)^*\text{-semi}$-generalised closures of distinct points are distinct.

**Proof** Let $x, y \in X$ and $x \neq y$. Case (i): $\{x\}$ is $(1,2)^*\text{-semi}$-closed. Then $\{x\}$ is $(1,2)^*\text{-sg}$-closed. Now $y \neq x$ implies $y \notin \{x\} = (1,2)^*\text{-sgcl}(\{x\})$. Hence $(1,2)^*\text{-sgcl}(\{y\}) \neq (1,2)^*\text{-sgcl}(\{x\})$.

Case (ii) $\{x\}$ is not $(1,2)^*\text{-semi}$-closed. Then $X - \{x\}$ not $(1,2)^*\text{-semi}$-open and therefore $X$ is the only $(1,2)^*\text{-semi}$-open set containing $X - \{x\}$. Hence $X - \{x\}$ is $(1,2)^*\text{-sg}$-closed. Now $y \in X - \{x\}$ implies $(1,2)^*\text{-sgcl}(\{y\}) \subseteq X - \{x\}$. Hence $x \notin (1,2)^*\text{-sgcl}(\{y\})$ and $(1,2)^*\text{-sgcl}(\{y\}) \neq (1,2)^*\text{-sgcl}(\{x\})$.

From Theorems 3.3 and 3.4 we conclude the following.
Theorem 3.5 Every bitopological space \((X, \tau_1, \tau_2)\) is \((1,2)^{-}\text{-sg-T}_0\).

Definition 3.6 A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1,2)^{-}\text{-semi-C}_0\) (resp. \((1,2)^{-}\alpha\text{-C}_0\)) space iff to each pair of distinct points \(x, y\) of \(X\), there exists a \((1,2)^{-}\text{-semi-open}\) (resp. \((1,2)^{-}\alpha\text{-open}\)) set such that \((1,2)^{-}\text{scl}(G)\) (resp. \((1,2)^{-}\alpha\text{cl}(G)\)) contains one of \(x\) and \(y\), but not the other.

Theorem 3.7 If a bitopological space \((X, \tau_1, \tau_2)\) is

(i) \((1,2)^{-}\alpha\text{-C}_0\) then it is \((1,2)^{-}\text{-semi-C}_0\).

(ii) \((1,2)^{-}\text{-semi-C}_0\) then it is \((1,2)^{-}\text{-semi-T}_0\).

Proof of (i) Let \((X, \tau_1, \tau_2)\) be \((1,2)^{-}\alpha\text{-C}_0\) and \(x, y \in X\) with \(x \neq y\). Then there exists a \((1,2)^{-}\text{-open}\) set \(G\) of \(X\) such that \(x \in (1,2)^{-}\text{cl}(G)\) and \(y \notin (1,2)^{-}\text{cl}(G)\). Since \(G\) is \((1,2)^{-}\text{-open}\), \(G_1 = (1,2)^{-}\text{cl}(G) = \tau_{1,2}\text{cl}(\tau_{1,2}\text{int}(G))\) (by Lemma 4.6[8]) is \((1,2)^{-}\text{-semi-open}\). Since \((1,2)^{-}\text{scl}(G_1) \subseteq (1,2)^{-}\text{cl}(G_1) = (1,2)^{-}\text{cl}(G)\), \(y \notin (1,2)^{-}\text{scl}(G_1)\). But \(x \in (1,2)^{-}\text{cl}(G) = G_1 \subseteq (1,2)^{-}\text{scl}(G_1)\). Hence \((X, \tau_1, \tau_2)\) is \((1,2)^{-}\text{-semi-C}_0\).

Proof of (ii) Let \((X, \tau_1, \tau_2)\) be \((1,2)^{-}\text{-semi-C}_0\) and \(x, y \in X\) with \(x \neq y\). Then there exists a \((1,2)^{-}\text{-semi-open}\) set \(G\) of \(X\) such that \(x \in (1,2)^{-}\text{scl}(G)\) and \(y \notin (1,2)^{-}\text{scl}(G)\). Since \(G\) is \((1,2)^{-}\text{-semi-open}\), \((1,2)^{-}\text{scl}(G)\) is \((1,2)^{-}\text{-semi-open}\). Also \(x \in (1,2)^{-}\text{scl}(G)\) but \(y \notin (1,2)^{-}\text{scl}(G)\). Hence \((X, \tau_1, \tau_2)\) is \((1,2)^{-}\text{-semi-T}_0\).

Remark 3.8 The converses in Theorem 3.7 are not true in general.

Definition 3.9 A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1,2)^{-}\text{-semi-generalised-T}_1\) (briefly written as \((1,2)^{-}\text{-sg-T}_1\)) space if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of \((1,2)^{-}\text{-sg-open}\) sets, one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).

Remark 3.10 A \((1,2)^{-}\text{-semi-T}_1\) bitopological space is \((1,2)^{-}\text{-sg-T}_1\) since every \((1,2)^{-}\text{-semi-open}\) set is \((1,2)^{-}\text{-sg-open}\).

We define the following.

Definition 3.11 A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called a \((1,2)^{-}\text{-semi-generalised neighbourhood}\) (briefly \((1,2)^{-}\text{-sgnbdl.}\)) of a point \(x\) of \(X\) if there exists a \((1,2)^{-}\text{-sg-open}\) set \(U\) containing \(x\) such that \(U \subseteq A\).

Definition 3.12 The union of all \((1,2)^{-}\text{-sg-open}\) sets of a bitopological space \((X, \tau_1, \tau_2)\) which are contained in a subset \(A\) of \(X\) is called the \((1,2)^{-}\text{-semi-generalised interior}\) of \(A\) and is denoted by \((1,2)^{-}\text{-sgint}(A)\).
Now we prove that arbitrary union of $(1, 2)^*\text{-slc}$ is $(1, 2)^*\text{-sg}$.

**Lemma 3.13** Every singleton set $\{x\}$ of a bitopological space $(X, \tau_1, \tau_2)$ is either $(1, 2)^*\text{-nowhere dense}$ or $(1, 2)^*\text{-preopen}.$

**Proof** Let $x \in X$. If $\{x\}$ is not $(1, 2)^*\text{-nowhere dense}$ then $G = \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}\{x\}) \neq \phi$. Suppose $x$ is not in $G$. Then $G^c$ contains $x$. Since $G^c$ is $(1, 2)^*\text{-closed}$, $G^c \supseteq \tau_{1,2}\text{-cl}\{x\} \supseteq G$ which implies $G = \phi$, a contradiction. Hence $\{x\} \subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}\{x\})$ or $\{x\}$ is $(1, 2)^*\text{-preopen}$.

Lemma 3.13 provides a decomposition $X = X_1 \cup X_2$ of $(X, \tau_1, \tau_2)$ where $X_1 = \{x \in X/\{x\} \text{ is } (1, 2)^*\text{-nowhere dense}\}$ and $X_2 = \{x \in X/\{x\} \text{ is } (1, 2)^*\text{-preopen}\}$. This decomposition is useful in proving the following result.

**Lemma 3.14** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset of $X$. Then

(i) $A$ is $(1, 2)^*\text{-sg}$ closed if and only if $X_1 \cap (1, 2)^*\text{-slc}(A) \subseteq A$.

(ii) $(1, 2)^*\text{-pcl}(A) \subseteq X_1 \cup A.$

**Proof** of (i) Let $A \subseteq X$ be $(1, 2)^*\text{-sg}$-closed and let $x \in X_1 \cap (1, 2)^*\text{-slc}(A)$. Now $\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}\{x\}) = \phi$ implies $\{x\}$ is $(1, 2)^*\text{-semi-closed}$. If $x$ is not in $A$ then $A \subseteq \{x\}^c$, a $(1, 2)^*\text{-semi-open}$ set. Since $A$ is $(1, 2)^*\text{-sg}$-closed, $(1, 2)^*\text{-slc}(A) \subseteq \{x\}^c$ which implies $x \notin (1, 2)^*\text{-slc}(A)$, a contradiction. Hence $x \in A$ and $X_1 \cap (1, 2)^*\text{-slc}(A) \subseteq A$. Conversely let $X_1 \cap (1, 2)^*\text{-slc}(A) \subseteq A$. Let $U$ be any $(1, 2)^*\text{-semi-open}$ set containing $A$. It is enough if we prove that $X_2 \cap (1, 2)^*\text{-slc}(A) \subseteq U$. Let $x \in X_2 \cap (1, 2)^*\text{-slc}(A)$, $\{x\}$ is $(1, 2)^*\text{-preopen}$ implies $\{x\} \subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}\{x\}) = G$. Suppose $x$ is not in $U$. Then $x$ is in $U^c$. Therefore $G = \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}\{x\}) \subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(U^c) \subseteq U^c$ since $U^c$ is $(1, 2)^*\text{-semi-closed}$. Then $G \cap U = \phi$ which implies $G \cap A = \phi$. This is a contradiction since, $x \in G$, a $(1, 2)^*\text{-semi-open}$ set and $x \in (1, 2)^*\text{-slc}(A)$ imply $G \cap A \neq \phi$.

Proof of (ii): Let $x \in (1, 2)^*\text{-pcl}(A)$. Suppose $x \notin X_1$. Then $\{x\}$ is $(1, 2)^*\text{-preopen}$ and thus $\{x\} \cap A \neq \phi$. This implies $x \in A$.

**Theorem 3.15** Arbitrary intersection of $(1, 2)^*\text{-sg}$-closed sets in a bitopological space is $(1, 2)^*\text{-sg}$-closed.

**Proof** Let $(A_i)_{i \in I}$ be any collection of $(1, 2)^*\text{-sg}$-closed subsets of the bitopological space $(X, \tau_1, \tau_2)$ and let $A = \cap_{i \in I} A_i$. It is enough if we prove that $X_1 \cap (1, 2)^*\text{-slc}(A) \subseteq A$. Let $x \in X_1 \cap (1, 2)^*\text{-slc}(A)$. Then $\{x\}$ is $(1, 2)^*\text{-nowhere dense}$. Then $\{x\}$ is $(1, 2)^*\text{-semi-closed}$. Suppose $x \notin A$. Then $x \notin A_i$ for some $i \in I$. Then $A_i \subseteq X - \{x\}$, a $(1, 2)^*\text{-semi-open}$ set. Since $A_i$ is $(1, 2)^*\text{-sg}$-closed, $(1, 2)^*\text{-slc}(A_i) \subseteq X - \{x\}$. This implies $x \notin (1, 2)^*\text{-slc}(A_i)$, a contradiction, since $x \in (1, 2)^*\text{-slc}(A)$ implies $x \in (1, 2)^*\text{-slc}(A_i)$ for every $i$. 
Remark 3.16 From Theorem 3.15 it follows that arbitrary union of $(1, 2)^*$-sg-open sets in a bitopological space is $(1, 2)^*$-sg-open. Hence $(1, 2)^*$-sgint$(A)$ is $(1, 2)^*$-sg-open.

The following lemma which can be easily proved.

Lemma 3.17 A subset of a bitopological space is $(1, 2)^*$-sg-open iff it is a $(1, 2)^*$-sgnbd of each of its points.

Definition 3.18 A point $x$ of $X$ is called a $(1, 2)^*$-semi-generalised interior point (briefly $(1, 2)^*$-sg-interior point) of $A$ iff $x \in (1, 2)^*$-sgint$(A)$.

The following lemma can also be easily proved.

Lemma 3.19 Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset of $X$. Then any point $x \in X$ is a $(1, 2)^*$-sg-interior point of $A$ iff $A$ is $(1, 2)^*$-sgnbd of $x$.

Theorem 3.20 For a bitopological space $(X, \tau_1, \tau_2)$ the following are equivalent.

(i) $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-sg-$T_1$.

(ii) Each one point set is $(1, 2)^*$-sg-closed in $X$.

Proof (i) $\Rightarrow$ (ii) Let $(X, \tau_1, \tau_2)$ be $(1, 2)^*$-sg-$T_1$ and $x \in X$. Suppose $(1, 2)^*$-sgcl$(\{x\}) \neq \{x\}$. Then we can find an element $y \in (1, 2)^*$-sgcl$(\{x\})$ with $y \neq x$. Since $X$ is $(1, 2)^*$-sg-$T_1$, there exist $(1, 2)^*$-sg-open sets $U$ and $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Now $x \in V^c$ and $V^c$ is $(1, 2)^*$-sg-closed. Therefore $(1, 2)^*$-sgcl$(\{x\}) \subseteq V^c$ which implies $y \in V^c$, contradiction. Hence $(1, 2)^*$-sgcl$(\{x\}) = \{x\}$ or $\{x\}$ is $(1, 2)^*$-sg-closed.

(ii) $\Rightarrow$ (i) Let $x, y \in X$ with $x \neq y$. Then $\{x\}$ and $\{y\}$ are $(1, 2)^*$-sg-closed. Therefore $U = (\{x\})^c$ and $V = (\{y\})^c$ are $(1, 2)^*$-sg-open and $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence $(X, \tau_1, \tau_2)$ is $(1, 2)^*$-sg-$T_1$.

Definition 3.21 A point $x$ of $X$ is called a $(1, 2)^*$-sg-limit point of a subset $A$ of $X$ if and only if $A \cap (G - \{x\}) \neq \phi$ for every $(1, 2)^*$-sg-open set $G$ containing $x$.

The set of all $(1, 2)^*$-sg-limit points of a set $A$ of $X$ is denoted by $(1, 2)^*$-sgdl$(A)$.

Lemma 3.22 If $A$ is a subset of a space $X$ then $(1, 2)^*$-sgcl$(A) = A \cup (1, 2)^*$-sgdl$(A)$.

Proof Let $x \in (1, 2)^*$-sgcl$(A)$ and suppose $x \notin A$. Let $G$ be any $(1, 2)^*$-sg-open set containing $x$. If $G \cap A = \phi$ then $A \subseteq G^c$, a $(1, 2)^*$-sg-closed set and therefore $(1, 2)^*$-sgcl$(A) \subseteq G^c$. This implies $x \notin (1, 2)^*$-sgcl$(A)$, a contradiction. Hence $G \cap A \neq \phi$, in particular $A \cap (G - \{x\}) \neq \phi$ and therefore $x \in (1, 2)^*$-sgdl$A$. 

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Lemma 3.23 A point \( x \in (1,2)^*-sgcl(A) \) if and only if every \((1,2)^*-sg\)-open set containing \( x \) contains a point of \( A \).

**Proof** Follows from the above lemma and the definition of \((1,2)^*-sg\)-limit point.

If \( X \) is \((1,2)^*-sg-T_1 \) and if \( p \in (1,2)^*-sgcl(A) \) for some subset \( A \) of \( X \), then it is not necessary that every \((1,2)^*-sg\)-open set of \( p \) contains infinitely many points of \( A \).

Example 3.24 Let \( X = \{a, b, c, d\} \); \( \tau_1 = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, X\} \)

\( \tau_2 = \{\phi, \{c, d\}, X\} \); \( \tau_{1,2}\)-open sets = \( \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{c, d\}, X\} \)

\((1,2)^*-semi-open sets = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{c, d\}, \{a, b\}, X\} \)

\((1,2)^*-sg\)-open sets = \( \{\phi, \{a\}, \{a, c\}, \{a, c, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b\}, X\} \)

\((1,2)^*-sgcl(T_1) \). For the set \( A = \{a, c, d\} \), \( b \) is a \((1,2)^*-sg\)-limit point but the \((1,2)^*-sg\)-open set \( b \) contains only a finite number of points from \( A \).

Definition 3.25 A mapping \( f : X \rightarrow Y \) is said to be a \((1,2)^*-sg\)-irresolute mapping if \( f^{-1}(G) \) is \((1,2)^*-sg\)-open in \( X \) whenever \( G \) is \((1,2)^*-sg\)-open in \( Y \).

The following theorem can be easily proved.

Theorem 3.26 Let \( f : X \rightarrow Y \) be an injective and \((1,2)^*-sg\)-irresolute mapping. If \( Y \) is \((1,2)^*-sg-T_1 \) then so also is \( X \).

Definition 3.27 A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1,2)^*-semi-generalised-T_2 \) (briefly written as \((1,2)^*-sg-T_2\)) space if to each pair of distinct points \( x, y \) of \( X \), there exists a pair of disjoint \((1,2)^*-sg\)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

Remark 3.28 A \((1,2)^*-semi-T_2 \) bitopological space is \((1,2)^*-sg-T_2 \) since every \((1,2)^*-semi-open set is \((1,2)^*-sg\)-open.

The following theorem can be easily proved.

Theorem 3.29 Let \( f : X \rightarrow Y \) be an injective and \((1,2)^*-sg\)-irresolute mapping. If \( Y \) is \((1,2)^*-sg-T_2 \) then \( X \) is also \((1,2)^*-sg-T_2 \).

Definition 3.30 A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1,2)^*-semi-generalised-R_0 \) (briefly written as \((1,2)^*-sg-R_0\)) space if for each \((1,2)^*-sg\)-open set \( G \), \( x \in G \), implies \((1,2)^*-sgcl(x) \subseteq G \).
Definition 3.31 A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-semi-generalised-\(R_1\) (briefly written as \((1, 2)^*\)-\(sg-R_1\)) space if for \(x, y \in X\) with \((1, 2)^*\)-\(sgcl\)(\(x\)) \(\neq\) \((1, 2)^*\)-\(sgcl\)(\(y\)), there exists a pair of disjoint \((1, 2)^*\)-\(sg\)-open sets \(U\) and \(V\) such that \((1, 2)^*\)-\(sgcl\)(\(x\)) \(\subseteq\) \(U\) and \((1, 2)^*\)-\(sgcl\)(\(y\)) \(\subseteq\) \(V\).

Theorem 3.32 If a bitopological space \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(sg-R_1\) then every singleton set is \((1, 2)^*\)-\(sg\)-closed.

Proof Suppose \(\{x\}\) is not \((1, 2)^*\)-\(sg\)-closed. Then \(x \neq y \in (1, 2)^*\)-\(sgcl\)(\(\{x\}\)). Then \((1, 2)^*\)-\(sgcl\)(\(\{y\}\)) \(\subseteq\) \((1, 2)^*\)-\(sgcl\)(\(\{x\}\)) and \((1, 2)^*\)-\(sgcl\)(\(\{y\}\)) \(\neq\) \((1, 2)^*\)-\(sgcl\)(\(\{x\}\)). Then \((1, 2)^*\)-\(sgcl\)(\(\{y\}\)) and \((1, 2)^*\)-\(sgcl\)(\(\{x\}\)) cannot be separated by disjoint \((1, 2)^*\)-\(sg\)-open sets, a contradiction since \(X\) is \((1, 2)^*\)-\(sg-R_1\).

Corollary 3.33 If a bitopological space \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(sg-R_1\) then it is \((1, 2)^*\)-\(sg-R_0\).

Theorem 3.34 The following are equivalent in a bitopological space \((X, \tau_1, \tau_2)\).

(i) \(X\) is \((1, 2)^*\)-\(sg-T_2\).

(ii) \(X\) is \((1, 2)^*\)-\(sg-R_1\) and \((1, 2)^*\)-\(sg-T_1\).

(iii) \(X\) is \((1, 2)^*\)-\(sg-R_1\) and \((1, 2)^*\)-\(sg-T_0\).

Proof of (i) \(\Rightarrow\) (ii) \(X\) is \((1, 2)^*\)-\(sg-T_2\) implies \(X\) is \((1, 2)^*\)-\(sg-T_1\) and therefore by Theorem 3.20, every singleton set in \(X\) is \((1, 2)^*\)-\(sg\)-closed. Let \(x, y \in X\) and \(x \neq y\). Since \(X\) is \((1, 2)^*\)-\(sg-T_2\), there exist two disjoint \((1, 2)^*\)-\(sg\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively. Since \(\{x\}\) and \(\{y\}\) are \((1, 2)^*\)-\(sg\)-closed, \(X\) is \((1, 2)^*\)-\(sg-R_1\).

Proof of (ii) \(\Rightarrow\) (iii) is obvious since \(X\) is \((1, 2)^*\)-\(sg-T_1\) implies \(X\) is \((1, 2)^*\)-\(sg-T_0\).

Proof of (iii) \(\Rightarrow\) (i) \(X\) is \((1, 2)^*\)-\(sg-R_1\) implies \(\{x\}\) is \((1, 2)^*\)-\(sg\)-closed for every \(x \in X\). Also \(x \neq y\) implies \((1, 2)^*\)-\(sgcl\)(\(\{x\}\)) \(\neq\) \((1, 2)^*\)-\(sgcl\)(\(\{y\}\)). Hence \(X\) is \((1, 2)^*\)-\(sg-R_1\) implies \(X\) is \((1, 2)^*\)-\(sg-T_2\).

4 \((1, 2)^*\)-\(\psi\) Separation Axioms

In this section, we define and study some new separation axioms by defining \((1, 2)^*\)-\(\psi\)-open sets which are stronger than \((1, 2)^*\)-semi-generalised separation axioms.

Definition 4.1 A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((1, 2)^*\)-\(\psi\)-closed if \((1, 2)^*\)-\(scl\)(\(A\)) \(\subseteq\) \(U\) whenever \(A \subseteq U\) and \(U\) is \((1, 2)^*\)-\(sg\)-open.

A set \(A\) is called \((1, 2)^*\)-\(\psi\)-open if \(A^c\) is \((1, 2)^*\)-\(\psi\)-closed.
Remark 4.2 If \( A \) is \((1, 2)^*\)-\(\psi\)-closed and \( U \) is \((1, 2)^*\)-sg-open with \( A \subseteq U \), then \((1, 2)^*\)-\(scl(A) \subseteq (1, 2)^*\)-\(sint(U) \). This follows from the definitions of \((1, 2)^*\)-\(\psi\)-closed set and \((1, 2)^*\)-sg-open set.

Theorem 4.3 (i) Every \((1, 2)^*\)-semi-closed set, and thus every \(\tau_{1,2}\)-closed set and every \((1, 2)^*\)-\(\alpha\)-closed set is \((1, 2)^*\)-\(\psi\)-closed.

(ii) Every \((1, 2)^*\)-\(\psi\)-closed set is \((1, 2)^*\)-sg-closed, and thus \((1, 2)^*\)-semi-preclosed and also \((1, 2)^*\)-gs-closed.

Proof follows from the definitions.

The following examples show that these implications are not reversible.

Example 4.4 Let \( X = \{a, b, c, d\} \); \( \tau_1 = \{\emptyset, \{a, b\}, X\} \)
\( \tau_2 = \{\emptyset, \{a, c\}, X\} \); \( \tau_{1,2}\)-open sets = \(\{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \)
\( A = \{b, c, d\} \) is \((1, 2)^*\)-\(\psi\)-closed but not \((1, 2)^*\)-semi-closed.

Example 4.5 Let \( X = \{a, b, c, d, e\} \); \( \tau_1 = \{\emptyset, \{a, d, e\}, \{b, c\}, X\} \)
\( \tau_2 = \{\emptyset, \{a, c, d\}, X\} \); \( \tau_{1,2}\)-open sets = \(\{\emptyset, \{a, d, e\}, \{b, c\}, \{b, c, d\}, X\} \)
\( A = \{b\} \) is \((1, 2)^*\)-sg-open and \((1, 2)^*\)-sg-closed. Since \((1, 2)^*\)-\(scl(A) = \{b, c\}, A \) is not \((1, 2)^*\)-\(\psi\)-closed.

Thus the class of \((1, 2)^*\)-\(\psi\)-closed sets properly contains the class of \((1, 2)^*\)-semi-closed sets, and thus properly contains the class of \((1, 2)^*\)-\(\alpha\)-closed sets and also properly contains the class of \(\tau_{1,2}\)-closed sets. Also the class of \((1, 2)^*\)-\(\psi\)-closed sets is properly contained in the class of \((1, 2)^*\)-sg-closed sets and hence it is properly contained in the class of \((1, 2)^*\)-semi-preclosed sets and contained in the class of \((1, 2)^*\)-gs-closed sets.

We define the following.

Definition 4.6 A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-neighbourhood of a point \( x \) of \( X \) if there exists a \((1, 2)^*\)-\(\psi\)-open set \( U \) containing \( x \) such that \( U \subseteq A \).

Definition 4.7 The union of all \((1, 2)^*\)-\(\psi\)-open sets of a bitopological space \((X, \tau_1, \tau_2)\) which are contained in a subset \( A \) of \( X \) is called the \((1, 2)^*\)-\(\psi\)-interior of \( A \) and is denoted by \((1, 2)^*\)-\(\psi\)-\(\text{int}(A)\).

Now we state the following lemma which can be easily proved.

Lemma 4.8 A subset of a bitopological space is \((1, 2)^*\)-\(\psi\)-open iff it is a \((1, 2)^*\)-\(\psi\)-nbd of each of its points.

Definition 4.9 A point \( x \) of \( X \) is called a \((1, 2)^*\)-\(\psi\)-interior point of \( A \) iff \( x \in (1, 2)^*\)-\(\psi\)-\(\text{int}(A)\).
The following lemma can also be easily proved.

**Lemma 4.10** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) be a subset of \(X\). Then any point \(x \in X\) is a \((1, 2)^*\)-\(\psi\)-interior point of \(A\) iff \(A\) is a \((1, 2)^*\)-\(\psi\)-nbd of \(x\).

**Definition 4.11** The \((1, 2)^*\)-\(\psi\)-closure of a subset \(A\) of \(X\) is the intersection of all \((1, 2)^*\)-\(\psi\)-closed sets that contains \(A\) and is denoted by \((1, 2)^*\)-\(\psi\)-cl\((A)\).

**Definition 4.12** A point \(x\) of \(X\) is called a \((1, 2)^*\)-\(\psi\)-limit point of a subset \(A\) of \(X\) if and only if \(A \cap (G - \{x\}) \neq \emptyset\) for every \((1, 2)^*\)-\(\psi\)-open set \(G\) containing \(x\).

**Definition 4.13** The set of all \((1, 2)^*\)-\(\psi\)-limit points of a set \(A\) of \(X\) is denoted by \((1, 2)^*\)-\(\psi\)-d\((A)\), is called \((1, 2)^*\)-\(\psi\)-derived set of \(A\).

**Definition 4.14** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-\(T_0\) space iff to each pair of distinct points \(x, y\) of \(X\), there exists a \((1, 2)^*\)-\(\psi\)-open set containing one but not the other.

Clearly, every \((1, 2)^*\)-semi-\(T_0\) space is \((1, 2)^*\)-\(\psi\)-\(T_0\) and every \((1, 2)^*\)-\(\psi\)-\(T_0\) space is \((1, 2)^*\)-\(sg\)-\(T_0\) since every \((1, 2)^*\)-semi-open set is \((1, 2)^*\)-\(\psi\)-open and every \((1, 2)^*\)-\(\psi\)-open set is \((1, 2)^*\)-\(sg\)-open. The converse is not true in general.

**Theorem 4.15** If in any bitopological space, \((X, \tau_1, \tau_2)\), \((1, 2)^*\)-\(\psi\)-closures of distinct points are distinct, then \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(\psi\)-\(T_0\).

**Proof** Let \(x, y \in X\) and \(x \neq y\). By the hypothesis, \((1, 2)^*\)-\(\psi\)cl\((x)\) \neq (1, 2)^*\)-\(\psi\)cl\((y)\). Then there exists a point \(z \in X\) such that \(z\) belongs to exactly one of the two sets, say \((1, 2)^*\)-\(\psi\)cl\((y)\) but not to \((1, 2)^*\)-\(\psi\)cl\((x)\). If \(y \in (1, 2)^*\)-\(\psi\)cl\((x)\), then \((1, 2)^*\)-\(\psi\)cl\((y)\) \subseteq (1, 2)^*\)-\(\psi\)cl\((x)\) which implies \(z \in (1, 2)^*\)-\(\psi\)cl\((x)\), a contradiction. So \(y \in X - (1, 2)^*\)-\(\psi\)cl\((x)\), a \((1, 2)^*\)-\(\psi\)-open set which does not contain \(x\). This shows that \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(\psi\)-\(T_0\).

**Definition 4.16** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-\(T_1\) space if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of \((1, 2)^*\)-\(\psi\)-open sets, one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).

**Remark 4.17** Every \((1, 2)^*\)-semi-\(T_1\) bitopological space is \((1, 2)^*\)-\(\psi\)-\(T_1\) and every \((1, 2)^*\)-\(\psi\)-\(T_1\) bitopological space is \((1, 2)^*\)-\(sg\)-\(T_1\) since every \((1, 2)^*\)-semi-open set is \((1, 2)^*\)-\(\psi\)-open and every \((1, 2)^*\)-\(\psi\)-open set is \((1, 2)^*\)-\(sg\)-open.

**Theorem 4.18** For a bitopological space \((X, \tau_1, \tau_2)\) the following are equivalent.
(i) \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(\psi\)-\(T_1\).

(ii) Each one point set is \((1, 2)^*\)-\(\psi\)-closed in \(X\).

**Proof** (i) ⇒ (ii) Let \((X, \tau_1, \tau_2)\) be \((1, 2)^*\)-\(\psi\)-\(T_1\) and \(x \in X\). Suppose \((1, 2)^*\)-\(\text{sgcl}\)\(\{x\}\) ≠ \(\{x\}\). Then we can find an element \(y \in (1, 2)^*\)-\(\text{cl}\)\(\{x\}\) with \(y \neq x\). Since \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_1\), there exist \((1, 2)^*\)-\(\text{sg}\)-open sets \(U\) and \(V\) such that \(x \in U\), \(y \notin U\) and \(y \in V\), \(x \notin V\). Now \(x \in V^c\) and \(V^c\) is \((1, 2)^*\)-\(\psi\)-closed. Therefore \((1, 2)^*\)-\(\text{cl}\)\(\{x\}\) ⊆ \(V^c\) which implies \(y \in V^c\), a contradiction. Hence \((1, 2)^*\)-\(\text{cl}\)\(\{x\}\) = \(\{x\}\) or \(\{x\}\) is \((1, 2)^*\)-\(\psi\)-closed.

(ii) ⇒ (i) Let \(x, y \in X\) with \(x \neq y\). Then \(\{x\}\) and \(\{y\}\) are \((1, 2)^*\)-\(\psi\)-closed. Therefore \(U = \{y\}^c\) and \(V = \{x\}^c\) are \((1, 2)^*\)-\(\psi\)-open and \(x \in U\), \(y \notin U\) and \(y \in V\), \(x \notin V\). Hence \((X, \tau_1, \tau_2)\) is \((1, 2)^*\)-\(\psi\)-\(T_1\).

**Definition 4.19** A mapping \(f : X \to Y\) is said to be a \((1, 2)^*\)-\(\psi\)-irresolute mapping if \(f^{-1}(G)\) is \((1, 2)^*\)-\(\psi\)-open in \(X\) whenever \(G\) is \((1, 2)^*\)-\(\psi\)-open in \(Y\).

The following theorem can be easily proved.

**Theorem 4.20** Let \(f : X \to Y\) be an injective and \((1, 2)^*\)-\(\psi\)-irresolute mapping. If \(Y\) is \((1, 2)^*\)-\(\psi\)-\(T_1\) then \(X\) is also \((1, 2)^*\)-\(\psi\)-\(T_1\).

**Definition 4.21** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-\(T_2\) space if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of disjoint \((1, 2)^*\)-\(\psi\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).

**Remark 4.22** Every \((1, 2)^*\)-semi-\(T_2\) bitopological space is \((1, 2)^*\)-\(\psi\)-\(T_2\) and every \((1, 2)^*\)-\(\psi\)-\(T_2\) bitopological space is \((1, 2)^*\)-\(\text{sg}\)-\(T_2\) since every \((1, 2)^*\)-semi-open set is \((1, 2)^*\)-\(\psi\)-open and every \((1, 2)^*\)-\(\psi\)-open set is \((1, 2)^*\)-\(\text{sg}\)-open.

The following theorem can be easily proved.

**Theorem 4.23** Let \(f : X \to Y\) be an injective and \((1, 2)^*\)-\(\psi\)-irresolute mapping. If \(Y\) is \((1, 2)^*\)-\(\psi\)-\(T_2\) then \(X\) is also \((1, 2)^*\)-\(\psi\)-\(T_2\).

**Definition 4.24** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-\(R_0\) space if for each \((1, 2)^*\)-\(\psi\)-open set \(G\), \(x \in G\), implies \((1, 2)^*\)-\(\text{cl}\)\(\{x\}\) ⊆ \(G\).

**Definition 4.25** A bitopological space \((X, \tau_1, \tau_2)\) is called a \((1, 2)^*\)-\(\psi\)-\(R_1\) space if for \(x, y \in X\) with \((1, 2)^*\)-\(\text{cl}\)\(\{x\}\) ≠ \((1, 2)^*\)-\(\text{cl}\)\(\{y\}\), there exists a pair of disjoint \((1, 2)^*\)-\(\psi\)-open sets \(U\) and \(V\) such that \((1, 2)^*\)-\(\text{cl}\)\(\{x\}\) ⊆ \(U\) and \((1, 2)^*\)-\(\text{cl}\)\(\{y\}\) ⊆ \(V\).

**Theorem 4.26** Every \((1, 2)^*\)-\(\psi\)-\(R_1\) bitopological space is \((1, 2)^*\)-\(\psi\)-\(R_0\).

Theorem 4.17 The following are equivalent in a bitopological space \((X, \tau_1, \tau_2)\).

(i) \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_2\).

(ii) \(X\) is \((1, 2)^*\)-\(\psi\)-\(R_1\) and \((1, 2)^*\)-\(\psi\)-\(T_1\).

(iii) \(X\) is \((1, 2)^*\)-\(\psi\)-\(R_1\) and \((1, 2)^*\)-\(\psi\)-\(T_0\).

Proof of (i) \(\Rightarrow\) (ii) \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_2\) implies \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_1\) and therefore by Theorem 4.17, every singleton set in \(X\) is \((1, 2)^*\)-\(\psi\)-closed. Let \(x, y \in X\) and \(x \neq y\). Since \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_2\), there exist two disjoint \((1, 2)^*\)-\(\psi\)-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively. Since \(\{x\}\) and \(\{y\}\) are \((1, 2)^*\)-\(\psi\)-closed, \(X\) is \((1, 2)^*\)-\(\psi\)-\(R_1\).

Proof of (ii) \(\Rightarrow\) (iii) is obvious since \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_1\) implies \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_0\).

Proof of (iii) \(\Rightarrow\) (i) Let \(x, y \in X\) and \(x \neq y\). Case(i): \((1, 2)^*\)-\(\psi\)cl\(\{x\}\) \(\neq (1, 2)^*\)-\(\psi\)cl\(\{y\}\). Since \(X\) is \((1, 2)^*\)-\(\psi\)-\(R_1\), there exist two disjoint \((1, 2)^*\)-\(\psi\)-open sets \(U\) and \(V\) such that \(U \supseteq (1, 2)^*\)-\(\psi\)cl\(\{x\}\) and \(V \supseteq (1, 2)^*\)-\(\psi\)cl\(\{y\}\). Then \(x \in U\) and \(y \in V\). Case(ii): \((1, 2)^*\)-\(\psi\)cl\(\{x\}\) = \((1, 2)^*\)-\(\psi\)cl\(\{y\}\). Since \(x \neq y\) and \(X\) is \((1, 2)^*\)-\(\psi\)-\(T_0\), there exists a \((1, 2)^*\)-\(\psi\)-open set \(U\) containing \(x\) but not \(y\). Then \(y \in U^c\), a \((1, 2)^*\)-\(\psi\)-closed set. This implies \((1, 2)^*\)-\(\psi\)cl\(\{y\}\) \(\subseteq U^c\) and therefore \((1, 2)^*\)-\(\psi\)cl\(\{x\}\) \(\subseteq U^c\) or \(x \in U^c\) which is a contradiction. Hence case(ii) is not possible.

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