On Norm Estimate of Comutator between Subnormal Operators

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Abstract
For the usual norm for both operators $S$ and $R$, one always following estimate $\|SR - RS\| \leq 2\|S\|\|R\|$, in this paper we'll try to give an improved estimate in the case $S$ and $T$ are subnormal operators. Our paper generalizes the case already studied by Kittaneh [7] where $S$ and $R$ are normal operators in $B(H)$, some results are proved and some particular cases are given.

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1 Introduction

Let $B(H)$ be the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space $H$. In general if there exist two operators $S, R \in B(H)$ such that $[S, R] = SR - RS = C$, then $C$ is said to be a commutator.

Using the triangular inequality and multiplicity, the famous estimate of commutator $C$ is given by

$$\|C\| = \|ST - TS\| \leq 2\|S\|\|T\|$$

Kittaneh in [6] and [7] and other improved this estimate in different cases. We used the technics of operator theory for have a norme estimate of $C$ the
commutator between two subnormal operators $S$ and $R$ which have normal extensions $N_S$ and $N_R$ in another space $B(K)$ larger than $B(H)$, it well known: $A \in B(H)$, A normal if and only if $A^*A = AA^*$ where $A^*$ is the adjoint operator of the operator $A$.

Numeric simulations, mainly based on operator theory technics, have become mandatory for all research sectors connected to the applied physics. The most recent statistics show their use at all levels and in all applications (instrumentation, dosimetry, radiation protection, medical physics, regulation, and in the various engineering specialties. They often provide very precise information when experiments are impossible, difficult or expensive. They may constitute an alternate research line for several laboratories for science and technology.

In this work, we give a suitable extension for our subnormal operators $S$ and $R$, and the estimate of the norme $\|SR - RS\|$, some special cases is proposed and in the end some open problems connected with our present work.

2 Preliminaries

we consider the algebra of all bounded linear operators $B(H)$, when $H$ is a separable infinite dimensional complex Hilbert space, and $K$ is onether space larger than $H$ (ie. $H \subset K$).

Definition 2.1 Let $S \in B(H)$ be an operator, we say that $S$ is subnormal if and only if it admits a normal extension $N_S$ ie. there exists another space $K$ larger than $H$ such that.

1. $H \subset K$.
2. $N_S$ is normal on $K$.
3. $N_S|_H = S$ ( $N_Sf = Sf$, $\forall f \in H$).

We can always define $K$ such as $K = H \oplus H^\perp$, where $N_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix}$ $Q : H^\perp \to H$ and $T$ is define on $H^\perp$.

Lemma 2.2 For any subnormal operator $S \in B(H)$ we can put its normal extension on the form $N_S = \begin{pmatrix} S & (S^*S - SS^*)^{\frac{1}{2}} \\ 0 & T \end{pmatrix}$, where $T = Q^*S(Q^*)^{-1}$ and $Q = (S^*S - SS^*)^{\frac{1}{2}}$.

Proof.
Let $S \in B(H)$ be a subnormal operator and $N_S$ its normal extension.

$N_S$ is normal $\iff N^*_S N_S = N_S N^*_S$ \hspace{1cm} (1)

$N_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix} \iff N^*_S = \begin{pmatrix} S^* & 0 \\ Q^* & T^* \end{pmatrix}$ \hspace{1cm} (2)

we have

\[ N^*_S N_S = \begin{pmatrix} S^* & 0 \\ Q^* & T^* \end{pmatrix} \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix} = \begin{pmatrix} S^*S & S^*Q \\ Q^*S & Q^*Q + T^*T \end{pmatrix} \] \hspace{1cm} (3)

$N_S N^*_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix} \begin{pmatrix} S^* & 0 \\ Q^* & T^* \end{pmatrix} = \begin{pmatrix} SS^* + QQ^* & QT^* \\ TQ^* & TT^* \end{pmatrix}$ \hspace{1cm} (4)

Combining (1), (3) et (4) we obtain

\[
\begin{cases}
QQ^* = S^*S - SS^* \\
S^*Q = QT^* \\
Q^*Q + T^*T = TT^*
\end{cases}
\] \hspace{1cm} (5)

Hence

\[
\begin{cases}
|Q^*|^2 = S^*S - SS^* \\
S^*Q = QT^* \\
Q^*Q = TT^* - T^*T
\end{cases}
\] \hspace{1cm} (6)

where $|Q^*| = (QQ^*)^{\frac{1}{2}}$.

If we put $Q^*$ in its polar decomposition $Q^* = U|Q^*$, where $U$ is unitary operator, from (6) (first equality) we get

$Q^* = U(S^*S - SS^*)^{\frac{1}{2}}$

we can put $U = I$ the identity, and therefore $Q^* = (S^*S - SS^*)^{\frac{1}{2}}$, it is easy to see that $Q = (S^*S - SS^*)^{\frac{1}{2}}$.

From (6) (second equality), we obtain $T = Q^* S (Q^*)^{-1}$, hence

$N_S = \begin{pmatrix} S & (S^*S - SS^*)^{\frac{1}{2}} \\ 0 & T \end{pmatrix}$ \hspace{1cm} (7)

which is quite normal and the proof is completed.

Halmos in [5], problem.157 has proved that $\sigma(N_S) \subseteq \sigma(S)$ where $\sigma(S)$ and $\sigma(N_S)$ are the spectrum of $S$ and its normal extension $N_S$ respectively.

If $\rho(S)$ and $\rho(N_S)$ are the spectral radius of $S$ and its normal extension $N_S$
respectively, Halmos result we gives that $\rho(N_S) \leq \rho(S)$. If $N_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix}$, we can easily show that

$$\forall p \geq 1, \quad N_S^p = \left( \begin{array}{c} S^p \sum_{i=0}^{p-1} S^{p-1-i}QT^i \\ 0 \end{array} \right)$$

from where we obtain $\|N_S^p\| \geq \|S^p\|, \forall p \geq 1$, hence $\|N_S^p\|^{1/p} \geq \|S^p\|^{1/p}, \forall p \geq 1$. If we take the limit of two side of this inequality we obtain $\rho(N_S) \geq \rho(S)$, combining this result with Halmos result we get $\rho(N_S) = \rho(S)$.

### 3 Main Results

**Lemma 3.1** If $S$ is subnormal operator in $B(H)$, and $N_S$ is it’s normal extension then $\|N_S\| = \|S\|$.

**Theorem 3.2** Let $S$ and $R$ be two subnormal operators in $B(H)$, given by their complex representation $S = A + iC$ and $R = B + iD$ such that $a_1 \leq A \leq a_2$, $b_1 \leq B \leq b_2$, $a_1 \leq C \leq a_2$, $b_1 \leq D \leq b_2$. If we pose $U = [A, C]^{1/2}$ and $V = [B, D]^{1/2}$ such that $u \leq U \leq \frac{1}{3}u$, $v \leq V \leq \frac{1}{3}v$ where $0 < a_1 \leq u < \frac{2}{3}u \leq a_2$ and $0 < b_1 \leq v < \frac{2}{3}v \leq b_2$, then we have the estimate

$$\|SR - RS\| \leq \frac{1}{144}(16a_2 - 9a_1)(16b_2 - 9b_1).$$

**Remark 3.3** Under the conditions $\frac{a_2}{a_1} \geq e^2 + 1$ and $\frac{b_2}{b_1} \geq e^2 + 1$ we obtain a well estimate.

**Proof.**

If $S$ and $R$ be two subnormal operators in $B(H)$, according to lemme 2.2, we can write that $N_S$ and $N_R$ are respectively the normal extension of $S$ and $R$ on $K = H \oplus H^\perp$, where $N_S = \begin{pmatrix} S & (S^*S - SS^*)^{1/2} \\ 0 & ((S^*S - SS^*)^{1/2})^*S((S^*S - SS^*)^{1/2})^* \end{pmatrix}$ and $N_R = \begin{pmatrix} R & (R^*R - RR^*)^{1/2} \\ 0 & ((R^*R - RR^*)^{1/2})^*R((R^*R - RR^*)^{1/2})^* \end{pmatrix}$.

If we pose $Q_S = (S^*S - SS^*)^{1/2}$, $Q_R = (R^*R - RR^*)^{1/2}$, $T_S = Q_S^*S(Q_S^*)^{-1}$ and $T_R = Q_R^*R(Q_R^*)^{-1}$, we obtain $N_SN_R - N_RN_S = \begin{pmatrix} SR - RS & SQ_R + Q_SR & RQ_S - RQS & QT_R - T_RS \\ 0 & T_S & T_R & T_RT_R \end{pmatrix}$. Therefore

$$\|SR - RS\| \leq \|N_SN_R - N_RN_S\| \quad (8)$$

If we replace the entries of matrices $N_S$ and $N_R$ by their cartesian forms, we get $N_S = \begin{pmatrix} A + iC & (1 + i)U \\ 0 & U^{-1}AU + iU^{-1}CU \end{pmatrix}$ and $N_R = \begin{pmatrix} B + iD & (1 + i)V \\ 0 & V^{-1}BV + iV^{-1}DV \end{pmatrix}$.
where $U = [A, C]^\frac{1}{2}$ and $V = [B, D]^\frac{1}{2}$.

Then we can write that $N_S = A + iC$ and $N_R = B + iD$ where

$$A = \begin{pmatrix} A & U \\ 0 & U^{-1}AU \end{pmatrix}, \quad C = \begin{pmatrix} C & U \\ 0 & U^{-1}CU \end{pmatrix}$$

$$B = \begin{pmatrix} B & V \\ 0 & V^{-1}BV \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D & V \\ 0 & V^{-1}DV \end{pmatrix}.$$

Using the conditions on the operators $A$, $B$, $C$, $D$, $U$ and $V$ given in the theorem we obtain

$$\frac{3}{4}a_1 \leq A \leq \frac{4}{3}a_2,$$

$$\frac{3}{4}a_1 \leq C \leq \frac{4}{3}a_2,$$

$$\frac{3}{4}b_1 \leq B \leq \frac{4}{3}b_2 \quad \text{and} \quad \frac{3}{4}b_1 \leq D \leq \frac{4}{3}b_2.$$

Applying the Kittaneh estimate ([5], Theorem 3) to the operators $N_S$ and $N_R$ and combining with inequality (8) we obtain the desired estimate.

**Corollary 3.4** If the interval contains $U$ will be wider, then the conditions $\frac{a_2}{a_1} \geq e^2 + 1$ and $\frac{b_2}{b_1} \geq e^2 + 1$ will be improved.

### 4 Open problems

1. What about the estimate in the case if the operators $A$, $B$, $C$ and $D$ are contraction? this case has great importance in physics and it due to the contraction property.

2. Can there establish the best estimate for other classes of operators larger than this case (Hyponormal, Paranormal, Normaloid, ... etc.).

### References


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