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Differential Forms and its Applications

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Abstract

In the present paper we have used the Differential forms also known as exterior calculus of E.Cartan [1922] in Pullback calculations and proving the main theorems of advanced calculus i.e. Green, Gauss and Stoke's theorems. In particular a convenient test for a transformation to be canonical is given with examples based on differential forms, which is more suitable than tests given by Goldstein et.al.[2004].

Keywords: Differential form, canonical transformation, exterior derivative, wedge product

(1) Introduction

The calculus of differential forms, developed by E.Cartan [1922], is one of the most useful and fruitful analytic techniques in differential geometry. The catalogue of concepts that are unified and simplified by forms is astonishing : divergence and curl of three-dimensional Euclidean geometry, Green's theorem, Stoke's theorem and Gauss's theorem, the theory of integration on manifolds and much more. In this paper we have proved the Green, Gauss and Stokes' theorems using differential form.

Since, a system can be described by more than one set of generalized coordinates, say for the motion of a particle in a plane, we may use as generalized coordinates either the cartesian coordinates (x, y) or the plane polar coordinates (r, θ). Both choices are equally valid, but one out of these two sets may be convenient for the problem under consideration. A convenient choice of coordinates for the problem under consideration, is the set of those generalized coordinates which are *cyclic*. It

rarely happens that all generalized coordinates are cyclic. In the case of central forces neither x nor y is cyclic, while θ is the cyclic coordinate. The number of cyclic coordinates can depend upon the choice of generalized coordinates, and for each problem there may be one particular choice for which all coordinates are cyclic. If one can find this set, then the solution of problem becomes trivial. Since the obvious generalized coordinates suggested by the problem may not normally be cyclic, so we must transform one set of variables to some other set of variables that may be more suitable. Hence, to know which transformation is more suitable for the problem under consideration, one need to test whether the transformation is canonical or not. Here, we offer a convenient test for a transformation to be canonical based on differential form, which is relatively recent addition to the tools available to the applied mathematicians, for another treatment of the criterion see Goldstein et.al.[2004].
We require the following definitions in our investigations.

(2) Differential form

Let U be an open subset of R^n and let $AT_p(R^n, R)$ be the space of p-linear continuous alternating mappings from R^n to R. Then $AT_p(R^n, R)$ is a Banach space.

A mapping $\omega : U \to AT_p(R^n, R)$ is called a differential form of degree p. A differential form ω is said to be of class C^n if the mapping ω is of class C^n, where n is an integer and $0 \le n \le \infty$. In general, on a "manifold" with $u = (u_1, u_2, ..., u_n)$ as a local coordinate system, a differential form of degree p, or a p-form in short, can be written in the form

$$\omega = \sum_{1 \le i_1 < i_2 < ... < i_p \le n} f_{i_1, i_2, ..., i_p}(u_1, u_2, ..., u_n)\, du_{i_1} \wedge du_{i_2} \wedge ... \wedge du_{i_p} .$$

Here the symbol $\wedge$ is called the wedge product and $dx \wedge dy$ is read as "dx wedge dy". The wedge product, like the more familiar cross product of vector calculus, is anti-commutative:

$$dx \wedge dy = -dy \wedge dx \quad \text{and} \quad dx \wedge dx = 0 \tag{2.1}$$

Consider two differential forms, α of degree p and β of degree q, their exterior product or wedge product written as $\alpha \wedge \beta$ obeys the usual rules of algebra and for each $x \in U$ and $v_1, ..., v_{p+q} \in (R^n)^{p+q}$ defined as follows ,

$$(\alpha \wedge \beta)(x)(v_1, ..., v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma} (-1)^{\sigma} (\alpha \otimes \beta)(x)(v_{\sigma(1)}, ..., v_{\sigma(p+q)})$$

$$= \frac{1}{p!q!}\sum_{\sigma}(-1)^{\sigma}\alpha(x)(v_{\sigma(1)},\ldots,v_{\sigma(p)}).\beta(x)(v_{\sigma(p+1)},\ldots,v_{\sigma(p+q)})$$

where the sum is taken over all permutation $\sigma \in S_{p+q}$ of $\{1,\ldots,p+q\}$.

(2.1) Differential form in $\mathbf{R}$:-

In the study of differential forms on real line $\mathbf{R}$, there are two objects of our use :

0-forms: which are just scalar functions of x, i.e. $f(x)$.

1-forms: which are differential of the form $f(x)\,dx$.

(2.2) Differential form in $\mathbf{R}^2$:-

In the study of differential forms on the two dimensional plane, there are three objects of our use:

0-forms: which are just scalar functions of x and y, i.e. $f(x,y)$.

1-forms: which are differentials of the form $f(x,y)\,dx + g(x,y)\,dy$, and

2-forms: which are elements of area, written as $f(x,y)\,dx \wedge dy$.

(2.3) Differential form in $\mathbf{R}^3$:-

Let x^1, x^2 and x^3 be rectangular coordinates on R^3. Then the differential forms which are useful on R^3, are:

0-forms: which are scalar functions of the form $f(x^1, x^2, x^3)$.

1-forms: $f_1\,dx^1 + f_2\,dx^2 + f_3\,dx^3$

2-forms: $f_1\,dx^2 \wedge dx^3 + f_2\,dx^3 \wedge dx^1 + f_3\,dx^1 \wedge dx^2$

3-forms: $f\,dx^1 \wedge dx^2 \wedge dx^3$

where f, f_i's are scalar functions of x^1, x^2 and x^3. Here, it should be noted that no 4-forms or higher order forms are possible in R^3, since these require at least one repeated differential, which gives zero by virtue of (2.1).

(3) Main Results

(3.1). Green's Theorem: - Let C be a closed C^1 curve in R^2 and D be the interior of C. If $f(x,y)$ and $g(x,y)$ are both C^1 functions defined on D, then

$$\int_C f\,dx + g\,dy = \int_D \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) dx\,dy \qquad (3.1.1)$$

Using the differential form the above Green's theorem can be put in compact form as,

$$\int_s d\omega = \int_{\partial s} \omega$$

where ω is the differential 1-form on $S \subset R^2, d\omega$ is the exterior derivative of ω and ∂s represents its boundary (a closed curve in the plane).

Proof: - Consider a differential form ω, we may associate with it its exterior derivative $d\omega$. This is obtained by taking, for each term of the differential form ω, the usual differential of the scalar multiplier, and wedging the result into the rest of the term.

Let $\omega = f(x,y)\,dx + g(x,y)\,dy$ is a differential 1-form on $S \subset R^2$, then

$$\begin{aligned} d\omega &= d(f\,dx + g\,dy) \\ &= df \wedge dx + dg \wedge dy \\ &= (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \wedge dx + (\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy) \wedge dy \\ &= (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\,dx \wedge dy \qquad (3.1.2) \end{aligned}$$

Using (3.1.2), Green's theorem (3.1.1), can be written in compact form as,

$$\int_S d\omega = \int_{\partial S} \omega \qquad (3.1.3)$$

where S, represents a region of the plane and ∂S represents its boundary (a closed curve in the plane).

(3.2). Stoke's Theorem in R^3 :- Let $S \subset R^3$ be a smooth open surface bounded by a closed curve C. If $\vec{F}(x,y,z)$ be a continuous vector function which has continuous first partial derivatives in $S \subset R^3$, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds \qquad (3.2.1)$$

where $\hat{n}$ is the outward drawn unit normal vector to $S \subset R^3$.

Using the differential form the above Stoke's theorem can be put in compact form as,

$$\iint_S d\omega = \int_{\partial S} \omega \qquad (3.2.2)$$

where S, represents a surface in R^3 and ∂S represents its boundary (a closed curve in R^3).

Proof:-In order to obtain the Stoke's theorem from the integral theorem (3.1.3),

let $\omega = f_1\,dx^1 + f_2\,dx^2 + f_3\,dx^3$ (in vector notation it is $\vec{F} \cdot d\vec{r}$) (3.2.3)

be a differential 1-form in R^3, where f_i's are functions of x^1, x^2, x^3, then

$$\begin{aligned} d\omega &= df_1 \wedge dx^1 + df_2 \wedge dx^2 + df_3 \wedge dx^3 \\ &= (\frac{\partial f_1}{\partial x^1}dx^1 + \frac{\partial f_1}{\partial x^2}dx^2 + \frac{\partial f_1}{\partial x^3}dx^3) \wedge dx^1 + (\frac{\partial f_2}{\partial x^1}dx^1 + \frac{\partial f_2}{\partial x^2}dx^2 + \frac{\partial f_2}{\partial x^3}dx^3) \wedge dx^2 \end{aligned}$$

$$+(\frac{\partial f_3}{\partial x^1}dx^1+\frac{\partial f_3}{\partial x^2}dx^2+\frac{\partial f_3}{\partial x^3}dx^3)\wedge dx^3$$

$$=(\frac{\partial f_2}{\partial x^1}-\frac{\partial f_1}{\partial x^2})dx^1\wedge dx^2+(\frac{\partial f_3}{\partial x^2}-\frac{\partial f_2}{\partial x^3})dx^2\wedge dx^3+(\frac{\partial f_1}{\partial x^3}-\frac{\partial f_3}{\partial x^1})dx^3\wedge dx^1 \quad (3.2.4)$$

Equation (3.2.2) in vector notation is $(\nabla\times\vec{F})\cdot\hat{n}\,ds$

Hence, using (3.1.3), (3.2.3) and (3.2.4) we get the Stokes' theorem (3.2.2) in R^3.

(3.3). Gauss's Divergence Theorem in $\mathbf{R}^3$:- Let $V\subset R^3$ be a volume bounded by a closed smooth surface S. If $\vec{F}(x,y,z)$ be a continuous vector function which has continuous first partial derivatives in $V\subset R^3$, then

$$\iiint_V \nabla\cdot\vec{F}\,dV=\iint_S \vec{F}\cdot\hat{n}\,ds \quad (3.3.1)$$

where $\hat{n}$ is the outward drawn unit normal vector to surface S.

Using the differential form the above Gauss's divergence theorem can be put in compact form as,

$$\iiint_S d\omega=\iint_{\partial S}\omega \quad (3.3.2)$$

where S, represents the interior of a closed surface in R^3 and ∂S represents its boundary.

Proof: - In order to obtain the Gauss's divergence theorem from the integral theorem (3.1.3), taking

$$\omega= f_1\,dx^2\wedge dx^3+f_2\,dx^3\wedge dx^1+f_3\,dx^1\wedge dx^2 \quad (3.3.3)$$

(which in vector notation corresponds to $\vec{F}\cdot\hat{n}\,ds$) be a differential 2-form in R^3, where f_i's are functions of x^1,x^2,x^3, then

$$d\omega=(\frac{\partial f_1}{\partial x^1}+\frac{\partial f_2}{\partial x^2}+\frac{\partial f_3}{\partial x^3})dx^1\wedge dx^2\wedge dx^3 \quad (3.3.4)$$

(which in vector notation equals $\nabla\cdot\vec{F}\,dV$). $\nabla\cdot\vec{F}\,dV$

Hence, using equations (3.1.3), (3.3.3) and (3.3.4), we get the Divergence theorem (3.3.2) in R^3.

The above results can be generalized to n-dimension, as the equation (3.1.3) is also valid in higher dimensions and the exterior product of a $p-$form and a $q-$form is $(p+q)-$form and the exterior derivative of a $p-$form is a $(p+1)-$form.

(4) Pullback Calculations

Let f be a mapping from R^2 to R^2, sending polar coordinates of a point (r,θ) to cartesian coordinates (x,y), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta .$$

then

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

or
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

or
$$\frac{\partial f}{\partial r} = \frac{x}{r}\frac{\partial f}{\partial x} + \frac{y}{r}\frac{\partial f}{\partial y}$$

or
$$\frac{\partial}{\partial r} = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

Similarly,

$$\frac{\partial}{\partial\theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Since, $\frac{\partial}{\partial r}$ is not defined at origin, i.e. $\frac{\partial}{\partial r}$ is not a differentiable vector field on R^2. So, we take an open subset U of R^2 as $U = R^2 - \{(0,0)\}$ and say that $\frac{\partial}{\partial r}$ is a differentiable vector field on U. Thus $\frac{\partial}{\partial r}$ is a differentiable vector field on U defined by

$$(x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

Define a mapping, $f : R^+ \times R \to R^2 - \{(0,0)\}$

by $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$. (4.1)

Since, this function goes from polar coordinates (r,θ) to cartesian coordinates (x,y), where $x = r\cos\theta$, $y = r\sin\theta$.

So, $dx = d(r\cos\theta) = \cos\theta\, dr - r\sin\theta\, d\theta$

and $dy = d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta$.

Then we define a two-form on cartesian coordinates as $dx \wedge dy$. Hence,

$$dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$

$$= r\ dr \wedge d\theta .$$

Here, it should be noted that the right hand side of the identity $dx \wedge dy = r\ dr \wedge d\theta$, depends on the way f is defined, but this dependence is not indicated on the left hand side. Hence, we must distinguish $dx \wedge dy$ from $r\ dr \wedge d\theta$. The proper way to express the connection between $dx \wedge dy$ and $r\ dr \wedge d\theta$ is to say :

"$r\ dr \wedge d\theta$ is the Pullback of $dx \wedge dy$ via f".

and we write,

$$f^*(dx \wedge dy) = r\ dr \wedge d\theta .$$

(4.1) Pullback of Laplace equation in two dimension from cartesian to polar coordinates :-

Definition: - The Hodge-star operator is the operator which in n-dimensions takes the p - form to $(n-p)$ -forms. Let $P(x,y)$ and $Q(x,y)$ be two continuous functions in R^2, then

$$*(P\,dx + Q\,dy) = P\,dy - Q\,dx$$

$$*\ (P\,dx \wedge dy) = P$$

$$*1 = dx \wedge dy .$$

Consider a mapping, $f : R^+ \times R \to R^2 - \{(0,0)\}$

such that $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$. (4.1.1)

Now,
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$*(df) = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy$$

$$d*(df) = (\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})\,dx \wedge dy$$

$$*d*(df) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \qquad (4.1.2)$$

Again, since
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Hence,
$$dr = \frac{x}{r}dx + \frac{y}{r}dy \quad \text{and} \quad d\theta = -\frac{y}{r^2}dx + \frac{x}{r^2}dy$$

$$dr \wedge d\theta = (\frac{x}{r}dx + \frac{y}{r}dy) \wedge (-\frac{y}{r^2}dx + \frac{x}{r^2}dy) = \frac{1}{r}dx \wedge dy$$

Thus,
$$*(dr) = r\,d\theta\ , \quad *(d\theta) = -\frac{x}{r^2}dx - \frac{y}{r^2}dy = -\frac{1}{r}dr\ ,$$

$$*(dr \wedge d\theta) = \frac{1}{r}.$$

Now, $$*(df) = *(\frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta)$$

$$= (\frac{\partial f}{\partial r}(r\, d\theta) - \frac{1}{r}\frac{\partial f}{\partial \theta} dr)$$

$$d*(df) = r\frac{\partial^2 f}{\partial r^2} dr \wedge d\theta + \frac{\partial f}{\partial r} dr \wedge d\theta - \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2} d\theta \wedge dr$$

$$*d*(df) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \qquad (4.1.3)$$

From equation (4.1.2) and (4.1.3), we have

$$f^*(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

Where, f is the mapping defined by equation (4.1.1). Hence, the Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ reduces to $\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$.

(5) Canonical Transformations

Let (M, ω, H) be a Hamiltonian system, where, (M, ω) is the symplectic manifold called the phase space of the system having 2n-coordinates $\{q^i, p_i : i = 1, 2, ..., n\ ;\ q^i \in R, p_i \in R^*\}$, ω is the symplectic two-form and H is the Hamiltonian function on M.

A transformation of the independent coordinates and momenta, q^i, p_i, to a new set Q^i, P^i, with (invertible) equations of transformation

$$Q^i = Q^i(q^i, p_i)$$
$$P^i = P^i(q^i, p_i)$$

is called a canonical transformation if this produces the same Hamiltonian $H(q^i(Q^i, P^i), p_i(Q^i, P^i))$ in new coordinate system (Q^i, P^i).

There are a number of techniques to show that a given transformation is canonical, see Goldstein[2004]. In terms of differential form, we define a canonical transformation as one which leaves ω in the same form i.e. a transformation is canonical iff

$$dq^i \wedge dp_i = dQ^i \wedge dP^i$$

Examples of canonical transformations:-

(5.1). Consider the change of coordinates from old coordinates (q,p) to new coordinates (Q,P) defined by,

$$Q = \log(\frac{1}{q}\sin p) \qquad and \qquad P = q\cot p$$

Thus,
$$dQ \wedge dP = d(\log(\frac{1}{q}\sin p) \wedge d(q\cot p)$$
$$= \frac{q}{\sin p}(-\frac{\sin p}{q^2}dq + \frac{\cos p}{q}dp) \wedge (\cot p\, dq - q\,\mathrm{cosec}^2 p\, dp)$$
$$= \mathrm{cosec}^2 p\, dq \wedge dp + \cot^2 p\, dp \wedge dq$$
$$= dq \wedge dp\,.$$

Hence, the given transformation is canonical.

(5.2). Consider the change of coordinates from old coordinates (q,p) to new coordinates (Q,P) defined by,

$$Q = \sqrt{2p}\sin q \qquad , \qquad P = \sqrt{2p}\cos q$$

Thus,
$$dQ \wedge dP = d(\sqrt{2p}\sin q) \wedge d(\sqrt{2p}\cos q)$$

$$= (\sqrt{2p}\cos q\, dq + \frac{1}{\sqrt{2p}}\sin q\, dp) \wedge (-\sqrt{2p}\sin q\, dq + \frac{1}{\sqrt{2p}}\cos q\, dp)$$
$$= \cos^2 q\, dq \wedge dp - \sin^2 q\, dp \wedge dq$$
$$= dq \wedge dp\,.$$

Hence, the given transformation is canonical.

(5.3) Consider the change of coordinates from old coordinates (q^1, q^2, p_1, p_2) to new coordinates (Q^1, Q^2, P_1, P_2) defined by,

$$Q^1 = (q^1)^2 \ , \quad Q^2 = q^2 \sec p_2$$
$$P_1 = \frac{p_1 \cos p_2 - 2q^2}{2q^1 \cos p_2} \ , \ P_2 = \sin p_2 - 2q^1$$

Thus,

$$dQ^1 \wedge dP_1 + dQ^2 \wedge dP_2$$
$$= [(2q^1 dq^1) \wedge \frac{1}{(q^1)^2}\{\frac{1}{2}(q^1 dp_1 - p_1 dq^1) - q^1 \sec p_2 dq^2 - q^1 q^2 \sec p_2 \tan p_2 dp_2 + q^2 \sec p_2 dq^1\}]$$
$$+[(\sec p_2 dq^2 + q^2 \tan p_2 \sec p_2 dp_2) \wedge (\cos p_2 dp_2 - 2dq^1)]$$
$$= [(dq^1 \wedge dp_1) - (2\sec p_2 dq^1 \wedge dq^2) - (2q^2 \sec p_2 \tan p_2 dq^1 \wedge dp_2)]$$

$$+[(dq^2 \wedge dp_2) - (2\sec p_2 dq^2 \wedge dq^1) - (2q^2 \sec p_2 \tan p_2 dp_2 \wedge dq^1)]$$
$$= dq^1 \wedge dp_1 + dq^2 \wedge dp_2 .$$

Hence, the given transformation is canonical.

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