

A Note on Summability Factors for a Triangular Matrix

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Abstract

In this paper we obtain an absolute summability factor theorem for lower triangular matrices under weaker conditions.

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1 Introduction

A weighted mean matrix, denoted by (\bar{N}, p_n) , is a lower triangular matrix with entries $\frac{p_k}{P_n}$, where $\{p_k\}$ is non-negative sequence with $p_0 > 0$ and $P_n = \sum_{k=0}^n p_k$.

Bor [2] used sufficient conditions on a sequence $\{p_k\}$ and a sequence $\{\lambda_n\}$ for $\sum \left(\frac{a_n P_n \lambda_n}{n p_n}\right)$ to be $|\bar{N}, p_n|_k$ absolute summable of order $k \geq 1$ by weighted mean (\bar{N}, p_n) . Unfortunately, he used an inappropriate definition of absolute summability (see, e.g., [4]).

Let A be an infinite lower triangular matrix. We may associate with A two lower triangular matrices \bar{A} and \hat{A} , whose entries are defined by

$$\bar{a}_{nk} = \sum_{i=k}^n a_{ni}, \quad \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k}, \quad (1)$$

respectively.

Let A be an infinite matrix. The series $\sum a_k$ is said to be absolutely summable by A , of order $k \geq 1$, written as $|A|_k$, if

$$\sum_{k=0}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty, \quad (2)$$

where Δ is the forward difference operator and t_n denotes the n -th term of the matrix transform of the sequence $\{s_n\}$, where $s_n = \sum_{k=0}^n a_k$.

Thus

$$t_n = \sum_{k=0}^n a_{nk} s_k = \sum_{k=0}^n a_{nk} \sum_{\nu=0}^k a_\nu = \sum_{\nu=0}^n a_\nu \sum_{k=\nu}^n a_{nk} = \sum_{\nu=0}^n \bar{a}_{n\nu} a_\nu,$$

$$t_n - t_{n-1} = \sum_{\nu=0}^n \bar{a}_{n\nu} a_\nu - \sum_{\nu=0}^{n-1} \bar{a}_{n-1,\nu} a_\nu = \sum_{\nu=0}^n \hat{a}_{n\nu} a_\nu, \quad (3)$$

since $\bar{a}_{n-1,n} = 0$.

2 Main Result.

The aim of this paper is to obtain an absolute summability factor theorem for a lower triangular matrix under weaker conditions, and as a corollary we obtain the correct version of Bor [2]. For this we need the concept of almost increasing sequence, a positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants B and D such that $Bc_n \leq b_n \leq Dc_n$ (see [1]). Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from example $b_n = ne^{(-1)^n}$.

Now, we shall prove the following theorem.

Theorem 2.1 *Let A be a triangle with nonnegative entries satisfying*

$$\bar{a}_{n0} = 1, n = 0, 1, \dots, \quad (4)$$

$$a_{n-1,\nu} \geq a_{n\nu} \text{ for } n \geq \nu + 1, \quad (5)$$

$$na_{nn} \asymp O(1), \quad (6)$$

$$\Delta \left(\frac{1}{a_{nn}} \right) = O(1), \quad (7)$$

$$\sum_{\nu=0}^n a_{\nu\nu} |a_{n,\nu+1}| = O(a_{nn}). \quad (8)$$

If (X_n) is an almost increasing sequence and the sequence (λ_n) and (β_n) satisfy

$$|\Delta \lambda_n| \leq \beta_n, \quad (9)$$

$$\lim \beta_n = 0, \quad (10)$$

$$|\lambda_n| X_n = O(1), \quad (11)$$

$$\sum_{n=1}^{\infty} nX_n |\Delta\beta_n| < \infty, \tag{12}$$

$$T_n = \sum_{\nu=1}^n \left(\frac{|s_\nu|^k}{\nu} \right) = O(X_n), \tag{13}$$

then the series $\sum_{\nu=1}^n \frac{a_n \lambda_n}{na_{nn}}$ is summable $|A|_k$, $k \geq 1$.

We need the following Lemma for the proof of our Theorem.

Lemma 2.2 [3] *Under the conditions on (X_n) , (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (12) is satisfied*

$$nX_n\beta_n = O(1), \text{ as } n \rightarrow \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{15}$$

Since (X_n) is almost increasing, $X_n \geq X_0$, which is a positive constant. Hence condition (11) implies that λ_n is bounded. It also follows from (14) that $\beta_n = O\left(\frac{1}{n}\right)$, and hence that $\Delta\lambda_n = O\left(\frac{1}{n}\right)$ by condition (9).

Proof of Theorem 2.1. Let (T_n) denote the n-th term of the A-transform of the series $\Sigma \left(\frac{a_n \lambda_n}{na_{nn}} \right)$. Then we may write

$$T_n = \sum_{\nu=0}^n a_{n\nu} \sum_{i=0}^{\nu} \frac{a_i \lambda_i}{ia_{ii}} = \sum_{i=0}^m \frac{a_i \lambda_i}{ia_{ii}} \sum_{\nu=i}^n a_{n\nu} = \sum_{i=0}^n \bar{a}_{ni} \frac{a_i \lambda_i}{ia_{ii}}.$$

Thus

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=0}^n \bar{a}_{ni} \frac{a_i \lambda_i}{ia_{ii}} - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} \frac{a_i \lambda_i}{ia_{ii}} = \sum_{i=0}^n (\bar{a}_{ni} - \bar{a}_{n-1,i}) \frac{a_i \lambda_i}{ia_{ii}} = \sum_{i=0}^n \hat{a}_{ni} \frac{a_i \lambda_i}{ia_{ii}} \\ &= \sum_{i=0}^n \hat{a}_{ni} \frac{\lambda_i}{ia_{ii}} (s_i - s_{i-1}) = \sum_{i=0}^{n-1} \hat{a}_{ni} \frac{\lambda_i}{ia_{ii}} s_i + a_{nn} \frac{\lambda_n}{na_{nn}} s_n - \sum_{i=0}^n \hat{a}_{ni} \frac{\lambda_i s_{i-1}}{ia_{ii}} \\ &= \sum_{i=0}^{n-1} \hat{a}_{ni} \frac{\lambda_i}{ia_{ii}} s_i + a_{nn} \frac{\lambda_n}{na_{nn}} s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \frac{\lambda_{i+1} s_i}{(i+1)a_{i+1,i+1}} \\ &= \sum_{i=0}^n \left(\hat{a}_{ni} \frac{\lambda_i}{ia_{ii}} - \hat{a}_{n,i+1} \frac{\lambda_{i+1}}{(i+1)a_{i+1,i+1}} \right) s_i + a_{nn} \frac{\lambda_n}{na_{nn}} s_n. \end{aligned}$$

We may write

$$\begin{aligned} \frac{\hat{a}_{ni} \lambda_i}{ia_{ii}} - \frac{\hat{a}_{n,i+1} \lambda_{i+1}}{(i+1)a_{i+1,i+1}} &= \frac{\hat{a}_{ni} \lambda_i}{ia_{ii}} - \frac{\hat{a}_{n,i+1} \lambda_{i+1}}{(i+1)a_{i+1,i+1}} + \frac{\hat{a}_{n,i+1} \lambda_i}{(i+1)a_{i+1,i+1}} \\ &\quad - \frac{\hat{a}_{n,i+1} \lambda_i}{(i+1)a_{i+1,i+1}} \\ &= \Delta_i \left(\frac{\hat{a}_{ni}}{ia_{ii}} \right) \lambda_i + \frac{\hat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta(\lambda_i). \end{aligned}$$

Also we may write

$$\begin{aligned}\Delta_i \left(\frac{\hat{a}_{ni}}{ia_{ii}} \right) \lambda_i &= \frac{\hat{a}_{ni}}{ia_{ii}} \lambda_i - \frac{\hat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \lambda_i - \frac{\hat{a}_{n,i+1}}{ia_{ii}} \lambda_i + \frac{\hat{a}_{n,i+1}}{ia_{ii}} \lambda_i \\ &= \frac{\Delta_i(\hat{a}_{ni}) \lambda_i}{ia_{ii}} + \hat{a}_{n,i+1} \lambda_i \left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \right).\end{aligned}$$

Hence,

$$\begin{aligned}T_n - T_{n-1} &= \sum_{i=0}^{n-1} \frac{\Delta_i(\hat{a}_{ni}) \lambda_i}{ia_{ii}} s_i + \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_i \left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \right) s_i \\ &\quad + \sum_{i=0}^{n-1} \frac{\hat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta_i(\lambda_i) s_i + \frac{\lambda_n}{n} s_n \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \text{ say.}\end{aligned}$$

To complete the proof of the Theorem 2.1, it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \text{ for } r = 1, 2, 3, 4.$$

Using Hölder's inequality and (6),

$$\begin{aligned}I_1 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \frac{\Delta_i(\hat{a}_{ni})}{ia_{ii}} \lambda_i s_i \right|^k \\ &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni}) \lambda_i s_i| \right)^k \\ &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| |\lambda_i|^k |s_i|^k \right) \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| \right)^{k-1}.\end{aligned}$$

But using (5),

$$\Delta_i(\hat{a}_{ni}) = \hat{a}_{ni} - \hat{a}_{n,i+1} = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1} = a_{ni} - a_{n-1,i} \leq 0.$$

Thus using (4),

$$\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| = \sum_{i=0}^{n-1} |a_{n-1,i} - a_{ni}| = 1 - 1 + a_{nn} = a_{nn}.$$

From (11), it follows that $\lambda_n = O(1)$. Using (6), (9), (13), and property (15) of Lemma 2.2,

$$I_1 = O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| |\lambda_i|^k |s_i|^k$$

$$\begin{aligned}
 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \\
 &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\Delta_i(\hat{a}_{ni})| \\
 &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k a_{ii} \\
 &= |\lambda_0| |s_0|^k a_{00} + O(1) \sum_{i=1}^m \frac{|\lambda_i| |s_i|^k}{i} \\
 &= O(1) + O(1) \sum_{i=0}^m |\lambda_i| \left[\sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right] \\
 &= O(1) \left[\sum_{i=1}^m |\lambda_i| \sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{j=0}^{m-1} |\lambda_{j+1}| \sum_{r=1}^j \frac{|s_r|^k}{r} \right] \\
 &= O(1) \sum_{i=1}^{m-1} \Delta(|\lambda_i|) X_i + O(1) |\lambda_m| X_m \\
 &= O(1) \sum_{i=1}^m \beta_i X_i + O(1) |\lambda_m| X_m \\
 &= O(1), \text{ as } m \rightarrow \infty. \\
 I_2 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_i \Delta\left(\frac{1}{ia_{ii}}\right) s_i \right|^k \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i| \Delta\left(\frac{1}{ia_{ii}}\right) |s_i| \right\}^k
 \end{aligned}$$

Now

$$\begin{aligned}
 \Delta\left(\frac{1}{ia_{ii}}\right) &= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \\
 &= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} + \frac{1}{(i+1)a_{ii}} - \frac{1}{(i+1)a_{ii}} \\
 &= \frac{1}{(i+1)} \left(\frac{1}{a_{ii}} - \frac{1}{a_{i+1,i+1}} \right) + \frac{1}{a_{ii}} \left(\frac{1}{i} - \frac{1}{(i+1)} \right) \\
 &= \frac{1}{(i+1)} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right].
 \end{aligned}$$

Thus using (7) and (5),

$$\begin{aligned}
 \left| \Delta\left(\frac{1}{ia_{ii}}\right) \right| &= \left| \frac{1}{i+1} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right] \right| \leq \frac{1}{i+1} \left\{ \frac{a_{i+1,i+1} - a_{ii}}{a_{ii}a_{i+1,i+1}} + \frac{1}{ia_{ii}} \right\} \\
 &= \frac{1}{i+1} [O(1) + O(1)] = O(1).
 \end{aligned}$$

Hence, using Hölder's inequality, (8) and (6),

$$\begin{aligned}
I_2 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i| \frac{1}{i+1} |s_i| \right\}^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i| a_{ii} |s_i| \right\}^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i|^k a_{ii} |s_i|^k \right) \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| a_{ii} \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i|^k a_{ii} |s_i|^k \\
&= O(1) \sum_{i=0}^m |\lambda_i|^k a_{ii} |s_i|^k \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\hat{a}_{n,i+1}| \\
&= O(1) \sum_{i=0}^m |\lambda_i|^k a_{ii} |s_i|^k \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}|
\end{aligned}$$

From [5],

$$\sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \leq 1.$$

Hence,

$$\begin{aligned}
I_2 &= O(1) \sum_{i=1}^m |\lambda_i|^k a_{ii} |s_i|^k = O(1) \sum_{i=1}^m |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \frac{1}{i} \\
&= O(1) \sum_{i=1}^m |\lambda_i| \frac{|s_i|^k}{i} = O(1), \text{ as } m \rightarrow \infty,
\end{aligned}$$

as in the proof of I_1 .

Using (6), Hölder's inequality, and (8).

$$\begin{aligned}
I_3 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \frac{\hat{a}_{n,i+1} (\Delta \lambda_i) s_i}{(i+1) a_{i+1,i+1}} \right|^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta \lambda_i| |s_i| \right)^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \frac{a_{ii}}{a_{ii}} |\hat{a}_{n,i+1}| |\Delta \lambda_i| |s_i| \right)^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta \lambda_i|^k |s_i|^k \right) \left(\sum_{i=0}^{n-1} a_{ii} |\hat{a}_{n,i+1}| \right)^{k-1}
\end{aligned}$$

$$\begin{aligned}
 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \\
 &= O(1) \sum_{n=1}^{m+1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i|^k |s_i|^k \frac{1}{a_{ii}^k} a_{ii} \\
 &= O(1) \sum_{n=1}^{m+1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i|^k |s_i|^k \frac{1}{a_{ii}^k} a_{ii} \\
 &= O(1) \sum_{i=0}^m \frac{a_{ii}}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \\
 &= O(1) \sum_{i=0}^m \left(\frac{|\Delta\lambda_i|}{a_{ii}} \right)^{k-1} |\Delta\lambda_i| |s_i|^k \\
 &= O(1) \sum_{i=0}^m |\Delta\lambda_i| |s_i|^k = O(1) \sum_{i=0}^m \beta_i |s_i|^k.
 \end{aligned}$$

Since $|s_i|^k = i(T_i - T_{i-1})$ by (13), we have

$$I_3 = O(1) \sum_{i=1}^m i(T_i - T_{i-1}) \beta_i.$$

Using Abel’s transformation, (9) and (15),

$$\begin{aligned}
 I_3 &= O(1) \sum_{i=1}^{m-1} T_i \Delta(i\beta_i) + O(1) mT_n \beta_n \\
 &= O(1) \sum_{i=1}^{m-1} i|\Delta\beta_i| X_i + O(1) \sum_{i=1}^{m-1} X_i \beta_i + O(1) mX_n \beta_n = O(1).
 \end{aligned}$$

Using (11) and (13)

$$\begin{aligned}
 I_4 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \frac{s_n \lambda_n}{n} \right|^k = \sum_{n=1}^{m+1} |s_n|^k |\lambda_n|^k \frac{1}{n} \\
 &= \sum_{n=1}^{m+1} \frac{|s_n|^k}{n} |\lambda_n| |\lambda_n|^{k-1} = O(1), \text{ as } m \rightarrow \infty,
 \end{aligned}$$

as it is the proof of I_1 .

Corollary 2.3 *Let $\{p_n\}$ be a positive sequence such that $P_n = \sum_{k=0}^n p_k \rightarrow \infty$ and satisfies*

$$np_n = O(P_n), \tag{16}$$

$$\Delta \left(\frac{P_n}{p_n} \right) = O(1). \quad (17)$$

If (X_n) is an almost increasing sequence and the sequences (λ_n) and (β_n) are such that

$$|\Delta \lambda_n| \leq \beta_n \quad (18)$$

$$\beta_n \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (19)$$

$$|\lambda_n| X_n = O(1), \text{ as } n \rightarrow \infty, \quad (20)$$

$$\sum_{n=1}^{\infty} n X_n |\Delta \beta_n| < \infty, \quad (21)$$

$$T_n = \sum_{\nu=1}^n \frac{|s_\nu|^k}{\nu} = O(X_n), \quad (22)$$

then the series $\sum \left(\frac{a_n P_n \lambda_n}{n p_n} \right)$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Proof of corollary. Conditions (18)-(22) of Corollary 2.3 are, respectively, conditions (9)-(13) of Theorem 2.1.

Conditions (4), (5) and (8) of Theorem 2.1 are automatically satisfied for any weighted mean method. Condition (6) and (7) of Theorem 2.1 become, respectively, conditions (16) and (4) of Corollary 2.3.

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