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On Nörlund Summability Factors of Infinite Series

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Abstract

In the present paper, a general theorem concerning the $\varphi - |N, p_n|_k$ summability factors of infinite series, has been proved.

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1 Introduction.

Let (φ_n) be a sequence of positive real numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By (t_n), we denote the n-th (C, 1) means of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see[4])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty, \tag{1}$$

and it is said to be summable $\varphi - |C, 1|_k$, $k \geq 1$, if (see[8])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n|^k < \infty. \tag{2}$$

If we take $\varphi_n = n$, then $\varphi - |C, 1|_k$ summablility reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0, \ (n \geq 0). \tag{3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu, \tag{4}$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|$, if (see[6])

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty, \tag{5}$$

and it is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see[3])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{6}$$

The series $\sum a_n$ is said to be summable $\varphi - |N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{7}$$

If we take $\varphi_n = n$ and $p_n = 1$ then $\varphi - |N, p_n|_k$ summability becomes $|C, 1|_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

2 Known Results.

Concerning the $|C, 1|$ and $|N, p_n|$ summability Kishore [5] has proved the following theorem.

Theorem 2.1 *Let* $p_0 > 0$, $p_n \geq 0$ *and* (p_n) *be a non-increasing sequence. If* $\sum a_n$ *is summable* $|C, 1|$, *then the series* $\sum a_n P_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

Later on Ram [7] has proved the following theorem related to the absolute Nörlund summability factors of infinite series.

Theorem 2.2 *Let* (p_n) *be as in Theorem 2.1. If*

$$\sum_{\nu=1}^{n} \frac{1}{\nu} |s_\nu| = O(X_n), \ n \to \infty, \tag{8}$$

where (X_n) *is positive non-decreasing sequence and* (λ_n) *is a sequence such that*

$$\sum_{n=1}^{\infty} n \left|\Delta^2 \lambda_n\right| X_n < \infty, \tag{9}$$

$$|\lambda_n| X_n = O(1), \ \text{as } n \to \infty. \tag{10}$$

then the sereis $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

Bor [1] has proved Theorem 2.2 under weaker conditions in the following form.

Theorem 2.3 *Let* (p_n) *be a sequence as in Theorem 2.1. If*

$$\sum_{\nu=1}^{n} \frac{1}{\nu} |t_\nu| = O(X_n), \text{ as } n \to \infty, \tag{11}$$

where (t_n) *is the n-th* $(\mathrm{C},1)$ *mean of the sequence* (na_n)*, and the sequences* (λ_n)*,* (X_n) *are such that conditions (9)-(10) of Theorem 2.2 are satisfied, then the series* $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

It should be noted that condition (8) implies the condition (11), but the converse need not be true (see[1] for details). Quite recently Bor [2] generalized Theorem 2.3 for $|N, p_n|_k$ summability in the following form.

Theorem 2.4 *Let* (p_n) *be a sequence as in Theorem 2.1. If*

$$\sum_{\nu=1}^{n} \frac{1}{\nu} |t_\nu|^k = O(X_n), \text{ as } n \to \infty, \tag{12}$$

and the sequences (λ_n)*,* (X_n) *satisfy the conditions (9) and (10) of Theorem 2.2, then the series* $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|_k$*,* $k \geq 1$.

3 Main Result.

The aim of this paper is to generalize Theorem 2.4 for $\varphi - |N, p_n|_k$ summability. Now we shall prove the following theorem.

Theorem 3.1 *Let* (φ_n) *be a sequence of positive real numbers. Let* (p_n) *be as in Theorem 2.1. The sequences* (λ_n) *and* (X_n) *satisfy the conditions (9) and (10) of Theorem 2.2 and if*

$$\sum_{\nu=1}^{n} \frac{\varphi_\nu^{k-1}}{\nu^k} |t_\nu|^k = O(X_n), \text{ as } n \to \infty, \tag{13}$$

$$\sum_{n=\nu+1}^{m+1} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_\nu^{k-1}}{\nu^k}\right), \tag{14}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $\varphi - |N, p_n|_k$*,* $k \geq 1$.

It should be noted that if we take k = 1 and $\varphi_n = n$ in Theorem 3.1, then we get Theorem 2.3 because in this case condition (13) reduces to condition (11) and condition (14) reduces to

$$\sum_{n=\nu}^{m} \frac{1}{n^2} = O\left(\frac{1}{\nu}\right), \tag{15}$$

but this always holds. We need the following lemma for the proof of our theorem.

Lemma 3.2 [1] *Under the conditions of* (X_n) *and* (λ_n), *as taken in the statement of Theorem 2.2, the following conditions hold,*

$$nX_n\Delta\lambda_n = O\left(1\right),\ as\ n \to \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} \Delta\lambda_n X_n < \infty. \tag{17}$$

4 Proof of Theorem 3.1.

Let (T_n) be n-th $(C,1)$ mean of the sequence $(na_n\lambda_n)$, that is

$$T_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu.$$

Applying Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu \lambda_\nu = \frac{1}{n+1}\sum_{\nu=1}^{n-1} \Delta\lambda_\nu \sum_{r=0}^{\nu} r a_r + \frac{\lambda_n}{n+1}\sum_{r=0}^{n} r a_r \\ &= \frac{1}{n+1}\sum_{\nu=1}^{n-1} \left(\nu+1\right)\Delta\lambda_\nu t_\nu + \lambda_n t_n \\ &= T_{n,1} + T_{n,2},\ say. \end{aligned}$$

Since $|T_{n,1} + T_{n,2}|^k \le 2^k\left(|T_{n,1}|^k + |T_{n,2}|^k\right)$, in order to complete the proof of the Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k}\left|T_{n,r}\right|^k < \infty,\ for\ r = 1, 2.$$

Now, we have that

$$\sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^k}\left|T_{n,1}\right|^k = \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^k}\left\{\frac{1}{n+1}\sum_{\nu=1}^{n-1} \frac{(\nu+1)}{\nu}\nu\left|\Delta\lambda_\nu\right|\left|t_\nu\right|\right\}^k$$

$$\begin{aligned}
&= \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{2k}} \left\{ \sum_{\nu=1}^{n-1} \nu \left|\Delta\lambda_\nu\right| \left|t_\nu\right| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{k+1}} \sum_{\nu=1}^{n-1} \left(\nu \left|\Delta\lambda_\nu\right|\right)^k \left|t_\nu\right|^k \times \left\{ \frac{1}{n} \sum_{\nu=1}^{n-1} 1 \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \frac{\varphi_n^{k-1}}{n^{k+1}} \sum_{\nu=1}^{n-1} \left(\nu \left|\Delta\lambda_\nu\right|\right)^k \left|t_\nu\right|^k \\
&= O(1) \sum_{\nu=1}^{m} \left(\nu \left|\Delta\lambda_\nu\right|\right)^k \left|t_\nu\right|^k \sum_{n=\nu+1}^{m+1} \frac{\varphi_n^{k-1}}{n^{k+1}} \\
&= O(1) \sum_{\nu=1}^{m} \left(\nu \left|\Delta\lambda_\nu\right|\right) \left(\nu \left|\Delta\lambda_\nu\right|\right)^{k-1} \frac{\varphi_\nu^{k-1}}{\nu^k} \left|t_\nu\right|^k \\
&= O(1) \sum_{\nu=1}^{m} \left(\nu \left|\Delta\lambda_\nu\right|\right) \frac{\varphi_\nu^{k-1}}{\nu^k} \left|t_\nu\right|^k \\
&= O(1) \sum_{\nu=1}^{m-1} \Delta\left(\nu \left|\Delta\lambda_\nu\right|\right) \sum_{r=1}^{\nu} \frac{\varphi_r^{k-1}}{r^k} \left|t_r\right|^k \\
&\quad + O(1)\, m \left|\Delta\lambda_m\right| \sum_{\nu=1}^{m} \frac{\varphi_\nu^{k-1}}{\nu^k} \left|t_\nu\right|^k \\
&= O(1) \sum_{\nu=1}^{m-1} \Delta\left(\nu \left|\Delta\lambda_\nu\right|\right) X_\nu + O(1)\, m \left|\Delta\lambda_m\right| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \left| (\nu+1) \left|\Delta^2\lambda_\nu\right| - \left|\Delta\lambda_\nu\right| \right| X_\nu + O(1)\, m \left|\Delta\lambda_m\right| X_m \\
&= O(1) \sum_{\nu=1}^{m-1} \nu \left|\Delta^2\lambda_\nu\right| X_\nu + O(1) \sum_{\nu=1}^{m-1} \left|\Delta\lambda_\nu\right| X_\nu \\
&\quad + O(1)\, m \left|\Delta\lambda_m\right| X_m \\
&= O(1), \ \textit{as } m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem 3.1 and Lemma 3.2.

Again,

$$\begin{aligned}
\sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k} \left|T_{n,2}\right|^k &= \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k} \left|\lambda_n\right|^k \left|t_n\right|^k \\
&= \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k} \left|\lambda_n\right| \left|\lambda_n\right|^{k-1} \left|t_n\right|^k \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k} \left|\lambda_n\right| \left|t_n\right|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta\left|\lambda_n\right| \sum_{\nu=1}^{n} \frac{\varphi_\nu^{k-1}}{\nu^k} \left|t_\nu\right|^k + O(1) \left|\lambda_n\right| \sum_{n=1}^{m} \frac{\varphi_n^{k-1}}{n^k} \left|t_n\right|^k
\end{aligned}$$

$$\begin{aligned} &= O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|X_n + O(1)\left|\lambda_m\right|X_m \\ &= O(1)\,, \ as\ m\to\infty, \end{aligned}$$

by virtue of the hypotheses of the Theorem 3.1 and Lemma 3.2. Therefore, we get

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|T_{n,r}\right|^k<\infty,\ for\ r=1,2.$$

This completes the proof of Theorem 3.1.

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