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Spacelike Curves of Constant Breadth in Minkowski 4-Space

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Abstract

In this paper, the concepts concerning the space curves of constant breadth in Euclidean 4-space are transformed to Minkowski 4-space. We show that when a spacelike curve (C) is given, a spacelike curve (C^*) can be determined and the integral third curvature of a spacelike curve of constant breadth is obtained as $\int_C k_3 ds = 0$.

Mathematics Subject Classifications: 53C50, 53C40

Keywords: Minkowski 3-space, spacelike curve, constant breadth

1 INTRODUCTION

The importance of the role of curves of constant breadth in cam designs led some authors to a strong interest in the subject. So far many geometers have shown increased interest in those properties of plane convex curves. Two brief reviews of the most important publications on this subject have been published by D. J. Struik [13]. Also a number of interesting properties of plane curves of constant breadth are included in the works of Euler, [6] E. Barbier [2], W. Blaschke [3] and A. P. Mellish [11].

A space curve of constant breadth was obtained by M. Fujiwara [7], by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which $d(P, Q)$is constant. For such curves PQ is also normal at Q; the chords PQ from a one-sided surface. Such a curve lies on a surface of constant breadth. Furthermore, W. Blaschke defined the curve of constant breadth on the sphere [4]. Ö. Köse presented some concept for space curves of constant breadth in Euclidean 3-space [9] and differential equations characterizing space curves of constant breadth were obtained by M. Sezer [12]. Similar characterization of space curves of constant breadth in Euclidean 4-space were given by A. Mağden and Ö. Köse [10]. But all of these papers are in Euclidean 2, 3 or 4-spaces.

In this paper we will give the spacelike curves of constant breadth in Minkowski 4-space.

2 PRELIMINARIES

The Minkowski 4-space E_1^4 is the real vector space R^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4 . An arbitrary vector $\vec{v} = (v_1, v_2, v_3, v_4)$ in E_1^4 can have one of three Lorentzian causal characters; it can be spacelike if $g(\vec{v}, \vec{v}) > 0$ or $\vec{v} = 0$, timelike if $g(\vec{v}, \vec{v}) < 0$ and null (lightlike) if $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Also recall that the pseudo-norm of an arbitrary vector $\vec{v} \in E_1^4$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v}, \vec{v})|}$. Therefore $\vec{v}$ is a unit vector if $g(\vec{v}, \vec{v}) = \pm 1$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Next, vectors $\vec{v}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(\vec{v}, \vec{w}) = 0$.

Denote by $\{\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4 . For an arbitrary spacelike curve $\alpha(s)$ with spacelike principal normal $\vec{N}$ in the space E_1^4 , the following Frenet formulae are given in [8]:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -\varepsilon k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix},$$

where $g(\vec{B}_1, \vec{B}_1) = \varepsilon = \pm 1$, $g(\vec{B}_2, \vec{B}_2) = -\varepsilon$ and k_1 , k_2 and k_3 are first, second and third curvatures of the spacelike curve respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike first binormal B_1 and timelike second binormal B_2 . If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike first binormal B_1 and spacelike second binormal B_2.

3 SPACELIKE CURVES OF CONSTANT BREADTH

Let $\vec{\alpha} = \vec{\alpha}(s)$ be a simple closed spacelike curve in Minkowski 4-space E_1^4 with nonzero curvatures. We will denote this curve by (C) . The normal plane at every point P on the curve meets the curve at a single point. This curve has parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite directions at corresponding points P and P' of the curve. The distance $d_{PP'}$ between the tangents $\vec{T}$ and $\vec{T}^*$ becomes maximum somewhere for the points P and P'. The distance $d_{PP'}$ is called as the breadth of the curve (C) according to the point P, and the point P' is called as the conjugate of P.

Let now (C) and (C^*) be unit speed spacelike curves from the class C^5. Assume that the tangents of (C) and (C^*) which correspond to opposite points $\vec{\alpha}(s) = \alpha$ and $\vec{\alpha}^*(s) = \alpha^*$ of the curves (C) and (C^*), respectively, are parallel and in opposite directions. The direction vector of the point α^* can be represented by the equation

$$\vec{\alpha}^*(s) = \vec{\alpha}(s) + m_1(s)\vec{T}(s) + m_2(s)\vec{N}(s) + m_3(s)\vec{B}_1(s) + m_4(s)\vec{B}_2(s), \tag{1}$$

where α and α^* are opposite points and $\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2$ denote the Frenet frame of the spacelike curve (C) in Minkowski 4-space. Differentiating this equation with respect to s, which is arc length of (C) and using the Frenet formulae we obtain

$$\begin{aligned} \frac{d\vec{\alpha}^*}{ds} = \vec{T}^*\frac{ds^*}{ds} = \big(1 + \frac{dm_1}{ds} - m_2k_1\big)\vec{T} + \big(m_1k_1 + \frac{dm_2}{ds} - \varepsilon m_3k_2\big)\vec{N} \\ + \big(m_2k_2 + \frac{dm_3}{ds} + m_4k_3\big)\vec{B}_1 + \big(m_3k_3 + \frac{dm_4}{ds}\big)\vec{B}_2, \end{aligned} \tag{2}$$

where the spacelike vector $\vec{T}^*$ denotes the tangent of (C^*) at the point α^*. Since $\vec{T}^* = -\vec{T}$, from (2) we have

$$\begin{aligned} 1 + \frac{dm_1}{ds} - m_2k_1 &= -\frac{ds^*}{ds}, \\ m_1k_1 + \frac{dm_2}{ds} - \varepsilon m_3k_2 &= 0, \\ m_2k_2 + \frac{dm_3}{ds} + m_4k_3 &= 0, \\ m_3k_3 + \frac{dm_4}{ds} &= 0. \end{aligned} \tag{3}$$

It is well known that the first curvature of (C) is $\lim \Delta\varphi/\Delta s = d\varphi/ds = k_1$. Here φ is the angle between the tangent of the curve (C) at point $\alpha(s)$ and a given spacelike fixed direction. Hence (3) may be written as follows:

$$\begin{aligned} \frac{dm_1}{d\varphi} &= m_2 - f(\varphi), \\ \frac{dm_2}{d\varphi} &= -m_1 + \varepsilon m_3 \rho k_2, \\ \frac{dm_3}{d\varphi} &= -m_4 \rho k_3 - m_2 \rho k_2, \\ \frac{dm_4}{d\varphi} &= -m_3 \rho k_3. \end{aligned} \tag{4}$$

Here $f(\varphi) = \rho + \rho^*$ and $\rho = 1/k_1$, $\rho^* = 1/k_1^*$ denote the radii of curvatures at the points α and α^*, respectively. If m_2, m_3, m_4 and their derivatives are eliminated in equations (4) we obtain the following equation with respect to m_1 :

$$\begin{aligned} & am_1^{(4)} + (2a - abc)m_1^{'''} + (a'' + a - a'bc + e - d)m_1^{''} \\ & + (2a' - abc - gb + e')m_1^{'} + (a'' - a'bc - d)m_1 + af''' \\ & + (2a' - abc)f'' + (a'' - a'bc - e - d)f' + (e' - gb)f = 0, \end{aligned} \tag{5}$$

where

$$a = \frac{1}{\varepsilon \rho k_2},\ b = (\rho k_3)',\ c = \frac{1}{\rho k_3},\ d = \frac{\rho k_3^2}{\varepsilon k_2},\ e = \rho k_2,\ g = \frac{k_2}{k_3}$$

and the derivatives are with respect to φ . This equation is a characterization for α^* . If the distance between the opposite points of (C) and (C^*) is constant, then

$$\|\alpha^* - \alpha\|^2 = m_1^2 + m_2^2 + \varepsilon m_3^2 - \varepsilon m_4^2 = constant.$$

Differentiating the last equation with respect to φ we have

$$m_1 m_1^{'} + m_2 m_2^{'} + \varepsilon m_3 m_3^{'} - \varepsilon m_4 m_4^{'} = 0. \tag{6}$$

Using the system (4), equation (6) is reduced to

$$m_1(m_1^{'} - m_2) = 0. \tag{7}$$

Thus we write $m_1 = 0$ or $\frac{dm_1}{d\varphi} = m_2$. If $\frac{dm_1}{d\varphi} = m_2$ then from the first equation in (4) it is easily seen that $f(\varphi) = 0$. It means that (C^*) is translated by the constant vector

$$\vec{l} = m_1(s)\vec{T}(s) + m_2(s)\vec{N}(s) + m_3(s)\vec{B}_1(s) + m_4(s)\vec{B}_2(s) \tag{8}$$

of (C). In the case of $m_2 = 0$ and $m_1 = constant = k$ we can write from (4)

$$f(\varphi) = 0,\ m_3 = \frac{m_1}{\varepsilon\rho k_2},\ \frac{dm_3}{d\varphi} = -m_4\rho k_3,\ \frac{dm_4}{d\varphi} = -m_3\rho k_3. \tag{9}$$

The change of variable $t(\varphi) = \int_0^\varphi \rho k_3 d\varphi$ gives us

$$\frac{d^2m_3}{dt^2} - m_3 = 0.$$

General solution of this equation is

$$m_3 = A\cosh\int_0^\varphi \rho k_3 d\varphi + B\sinh\int_0^\varphi \rho k_3 d\varphi.$$

where A and B are arbitrary constants. Then from (9) we have

$$m_4 = -A\sinh\int_0^\varphi \rho k_3 d\varphi - B\cosh\int_0^\varphi \rho k_3 d\varphi.$$

Hence the general solution set of the system (9) is given by

$$\begin{aligned} &m_1 = k,\ m_2 = 0,\ m_3 = A\cosh\int_0^\varphi \rho k_3 d\varphi + B\sinh\int_0^\varphi \rho k_3 d\varphi, \\ &m_4 = -A\sinh\int_0^\varphi \rho k_3 d\varphi - B\cosh\int_0^\varphi \rho k_3 d\varphi. \end{aligned} \tag{10}$$

Therefore equation (1) becomes

$$\begin{aligned} \vec{\alpha}^*(s) = \vec{\alpha}(s) + k\vec{T}(s) + &\left(A\cosh\int_0^\varphi \rho k_3 d\varphi + B\sinh\int_0^\varphi \rho k_3 d\varphi\right)\vec{B}_1(s) \\ - &\left(A\sinh\int_0^\varphi \rho k_3 d\varphi + B\cosh\int_0^\varphi \rho k_3 d\varphi\right)\vec{B}_2(s). \end{aligned} \tag{11}$$

The distance between the opposite points of these curves is $\sqrt{|k^2 + \varepsilon(A^2 - B^2)|}$. Thus from (9)

$$m_3 = \frac{m_1}{\varepsilon\rho k_2} = \frac{kk_1}{\varepsilon k_2},$$

and so that we obtain

$$\frac{k_1}{k_2} = A_1\cosh\int_0^\varphi \rho k_3 d\varphi + B_1\sinh\int_0^\varphi \rho k_3 d\varphi, \tag{12}$$

where $A_1 = \frac{\varepsilon}{k}A$, $B_1 = \frac{\varepsilon}{k}B$.

In the case $m_1 = 0$, we write from (4)

$$\begin{aligned} &m_2 = f(\varphi) \neq 0, \\ &\frac{dm_2}{d\varphi} = \varepsilon m_3 \rho k_2, \\ &\frac{dm_3}{d\varphi} = -m_4 \rho k_3 - m_2 \rho k_2, \\ &\frac{dm_4}{d\varphi} = -m_3 \rho k_3. \end{aligned} \tag{13}$$

By taking $\lambda = \rho k_2$, $\mu = \rho k_3$ and $u = \int_0^{\varphi} \mu d\varphi$ and using system (13) we obtain the differential equation

$$\frac{d^2 m_3}{du^2} - m_3 = -\frac{d}{du}(\frac{\lambda}{\mu} m_2). \tag{14}$$

General solution of (14) is

$$m_3 = A_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi - \int_0^{\varphi} \cosh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt, \tag{15}$$

where A_2 and B_2 are arbitrary constants. From equations (13) and (15) we find

$$m_4 = -\left(A_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi \right) - \int_0^{\varphi} \sinh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt. \tag{16}$$

Therefore the general solution of system (9) is

$$\begin{aligned} &m_1 = 0, \\ &m_2 = f(\varphi), \\ &m_3 = A_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi - \int_0^{\varphi} \cosh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt, \\ &m_4 = -\left(A_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi \right) - \int_0^{\varphi} \sinh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt. \end{aligned} \tag{17}$$

Then by using the equations (1) and (17) the curve (C^*) can be found as follows

$$\begin{aligned} &\vec{\alpha}^* = \vec{\alpha} + f(\varphi) \vec{N} \\ &+ \left(A_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi - \int_0^{\varphi} \cosh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt \right) \vec{B}_1 \\ &- \left(A_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi + \int_0^{\varphi} \sinh[u(\varphi) - u(t)] \rho(t) k_2(t) f(t) dt \right) \vec{B}_2. \end{aligned} \tag{18}$$

Using the fact that the curve of constant breadth is simple closed curve we have $\vec{\alpha}^*(0) = \vec{\alpha}^*(2\pi)$. Thus we can give the following corollaries.

Corollary 3.1 *Let (C^*) be a spacelike curve in Minkowski 4-space with properties that $k_1 > 0$, k_2 and k_3 are continuous periodic functions. If (C^*) is a spacelike curve of constant breadth then*

$$\int_0^{2\pi} \rho k_3 d\varphi = 0. \tag{19}$$

Since $\rho d\varphi = ds$, (19) becomes

$$\int_0^{2\pi} k_3 ds = 0. \tag{20}$$

Corollary 3.2 *Let (C^*) be a spacelike curve of constant breadth in Minkowski 4-space. Then*

$$\begin{aligned} &\int_0^{\varphi} \cosh[u(\varphi) - u(t)]\rho(t)k_2(t)f(t)dt = 0, \\ &\int_0^{\varphi} \sinh[u(\varphi) - u(t)]\rho(t)k_2(t)f(t)dt = 0. \end{aligned} \tag{21}$$

If the function of the curvatures of the spacelike curve of constant breadth in Minkowski 4-space is defined by h , then from Corollary 3.2 we have

$$\int_0^{2\pi} h(k_1, k_2, k_3) ds = 0. \tag{22}$$

Using (22) and considering (18) we have

$$\begin{aligned} f^2 + \varepsilon\Big(A_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi\Big)^2 \\ - \varepsilon\Big(A_2 \sinh \int_0^{\varphi} \rho k_3 d\varphi + B_2 \cosh \int_0^{\varphi} \rho k_3 d\varphi\Big)^2 = r^2 = \mathit{constant}. \end{aligned}$$

So that for the spacelike curves of constant breadth we get

$$r = \sqrt{|f^2 + \varepsilon(A_2^2 - B_2^2)|}.$$

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