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M-Quasihyponormal Composition Operators on Weighted Hardy Spaces

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Abstract

If T is an analytic function mapping the unit disk D into itself, we define the composition operator C_T on the space $H^2(\beta)$ by $C_T f = f \circ T$. In this paper, we investigate the relationship between properties of the symbol T and the quasihyponormality of the operators C_T and C_T^*.

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1. PRELIMINARIES

Let f be an analytic map on the open unit disk D given by the Taylor's series.

$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots..$

Let $\beta = \{\beta_n\}_{n=0}^{\alpha}$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ as $n \to \infty$.

The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\| f \|_\beta^2 = \sum_{n=0}^{\infty} | a_n |^2 \beta_n^2 < \infty$$

is a Hilbert space of functions analytic in the unit disc with the inner product.

$$\langle f, g\rangle_\beta = \sum_{n=0}^{\infty} a_n \overline{b}_n \beta_n^2 \quad \text{for f as above and } g(z) = \sum_{n=0}^{\infty} b_n z^n .$$

Let D be the open unit disk in the complex plane and let $T: D \to D$ be an analytic self-map of the unit disk and consider the corresponding composition operator C_T acting on $H^2(\beta)$, i.e.,

$$C_T(f) = f \circ T, \qquad f \in H^2(\beta)$$

The operators C_T are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases: on the classical Hardy space H^2 (the case when $\beta_n = 1$ for all n). See for example [7], and on a general space $H^2(\beta)$ if the function T is analytic on some open set containing the closed unit disk having supremum norm strictly smaller than one (see [11]). There are a lot of other known properties of composition operators, on the classical Hardy space H^2 (See for example [1], [5] and [7]), and on more general space $H^2(\beta)$ (see [3], [4], [8], [10] and [11]).

In [2], Cowen's and Kriete obtained a nice correlation between hyponormality of composition operators on H^2 and the Denjoy-Wolff point of the induced map.

In [9], Nina Zorboska obtained some results on the hyponormality of composition operators and their adjoints.

In this article, we are interested in the M-quasihyponormality of composition operators and their adjoints.

2. An operator T on a Hilbert space H is called M-quasihyponormal if there exists M>0 such that

$$M^2 T^{*2} T^2 - (T^*T)^2 \geq 0$$

If M = 1, T is said to be quasihyponormal.

Furuta *et al.* [6], introduced a new class 'class A' operators as follows.

An operator T belongs to class A if and only if

$$(T^* | T |^2 T)^{½} \geq T^*T$$

and showed that this class is included in the class of paranormal operators.

Let ω be a point on the open disk.

Define

$$k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{Z^n \overline{\omega}^n}{\beta_n^2}$$

Then the function k_ω^β is a point evaluation for $H^2(\beta)$.

Then k_ω^β is in $H^2(\beta)$ and $\left\| k_\omega^\beta \right\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{\beta_n^2}$.

Thus, $\| K_\omega \|$ is an increasing function of $| \omega |$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\left\langle f, k_\omega^\beta \right\rangle_\beta = \sum_{n=0}^{\infty} \frac{a_n \omega^n \beta_n^2}{\beta_n^2}$$

$$= f(\omega)$$

Therefore,

$$\left\langle f, k_\omega^\beta \right\rangle_\beta = f(\omega) \qquad \text{for all f and}$$

k_ω^β is known as the point evaluation kernal at ω.

It can be easily shown that

$$C_T^* \; k_\omega^\beta = k_{T(\omega)}^\beta$$

and $k_0^\beta = 1$ (the function identically equal to 1).

Theorem 2.1:

If C_T is M-quasihyponormal then $\left\| k_{T(0)}^\beta \right\|_\beta^2 \leq M^2$

Proof:

C_T is M-quasihyponormal.

$$\left\langle M^2 C_T^{*2} C_T^2 f, f \right\rangle - \left\langle (C_T^* C_T)^2 f, f \right\rangle \geq 0, \quad \text{for all } f \in H^2(\beta).$$

$$M^2 \left\langle C_T^2 f, C_T^2 f \right\rangle - \left\langle C_T^* C_T C_T^* C_T f, f \right\rangle \geq 0$$

$$M^2 \left\langle C_T^2 f, C_T^2 f \right\rangle - \left\langle C_T^* C_T f, C_T^* C_T f \right\rangle \geq 0$$

$$M^2 \| C_T^2 f \|^2 \geq \| C_T^* C_T f \|^2$$

Let $f = k_0^\beta$, we have,

$$M^2 \| C_T C_T k_0^\beta \|_\beta^2 \geq \| C_T^* C_T k_0^\beta \|_\beta^2$$

$$M^2 \| C_T k_0^\beta \|_\beta^2 \geq \| C_T^* k_0^\beta \|_\beta^2$$

$$M^2 \| k_0^\beta \|_\beta^2 \geq \| k_{T(0)}^\beta \|_\beta^2$$

$$\| k_{T(0)}^\beta \|_\beta^2 \geq M^2. \quad [9]$$

Theorem 2.2 :

A partial isometry composition operator C_T on $H^2(\beta)$ is M-quasi-hyponormal then $M^2 \geq 1$.

Proof :

C_T is M-quasihyponormal.

$$\| C_T^* C_T f \|^2 \leq M^2 \| C_T^2 f \|, \quad \text{for all } f \in H^2(\beta)$$

$$M^2 C_T^{*2} C_T^2 - 2k(C_T^* C_T)^2 + k^2 \geq 0 \quad \text{for all } k > 0$$

$$(M^2 C_T^{*2} C_T^2 - 2k(C_T^* C_T)^2 + k^2) C_T^* C_T \geq 0$$

$$M^2 C_T^* C_T^* C_T C_T C_T^* C_T - 2k C_T^* C_T C_T^* C_T C_T^* C_T + k^2 C_T^* C_T \geq 0$$

$$M^2 C_T^* C_T^* C_T C_T - 2k C_T^* C_T C_T^* C_T + k^2 C_T^* C_T \geq 0$$

$$M^2 C_T^* C_T^* C_T C_T - 2k C_T^* C_T + k^2 C_T^* C_T \geq 0$$

$$M^2 \| C_T^2 f \|^2 - 2k \| C_T f \|^2 + k^2 \| C_T f \|^2 \geq 0$$

Let f = k_0^β, we have,

$$M^2 \| C_T^2 \; k_0^\beta \|_\beta^2 - 2k \| C_T \, k_0^\beta \|_\beta^2 + k^2 \| C_T \; k_0^\beta \|_\beta^2 \geq 0$$

$$M^2 \| C_T \, C_T \; k_0^\beta \|_\beta^2 - 2k \| \; k_0^\beta \|_\beta^2 + k^2 \| \; k_0^\beta \|_\beta^2 \geq 0$$

$$M^2 \| C_T \; k_0^\beta \|_\beta^2 - 2k + k^2 \geq 0$$

$$M^2 \| \; k_0^\beta \|_\beta^2 - 2k + k^2 \geq 0$$

$$M^2 - 2k + k^2 \geq 0$$

By elementary properties of real quadratic form, we get, $M^2 \geq 1$.

Theorem 2.3:

If C_T is hyponormal composition operator on $H^2(\beta)$ and C_T^* is M-quasihyponormal then $M^2 \geq 1$.

Proof :

C_T^* is M-quasihyponormal

$$\left\langle M^2 C_T^2 C_T^{*2} \, f, f \right\rangle - \left\langle (C_T C_T^*)^2 f, f \right\rangle \geq 0, \quad \text{for all } f \in H^2(\beta)$$

$$M^2 \left\langle C_T^2 \, C_T^{*2} \, f, f \right\rangle - \left\langle C_T \, C_T^* \, C_T \, C_T^* \; f, f \right\rangle \geq 0$$

$$M^2 \left\langle C_T^{*2} \, f, C_T^{*2} \; f \right\rangle - \left\langle C_T C_T^* f, \, C_T C_T^* f \right\rangle \geq 0$$

$$M^2 \| \, C_T^{*2} \; f \|^2 - \| \, C_T \, C_T^* \; f \|^2 \geq 0$$

Let f = k_0^β, we have,

$$M^2 \| \, C_T^{*2} \; k_0^\beta \|_\beta^2 - \| C_T \, C_T^* \; k_0^\beta \|_\beta^2 \geq 0$$

$$M^2 \| \, C_T^{*2} \; k_0^\beta \|_\beta^2 \geq \| C_T \, C_T^* \; k_0^\beta \|_\beta^2$$

$$M^2 \| \, C_T^{*} \; k_{T(0)}^\beta \|_\beta^2 \geq \| C_T \; k_{T(0)}^\beta \|_\beta^2$$

Since C_T is hyponormal, we have T(0) = 0. [9]

$$M^2 \| \, C_T^{*} \; k_0^\beta \|_\beta^2 \geq \| C_T \; k_0^\beta \|_\beta^2$$

$$M^2 \| k^{\beta}_{T(0)} \|^2_{\beta} \geq \| k^{\beta}_{0} \|^2_{\beta}$$
$$M^2 \| k^{\beta}_{0} \|^2_{\beta} \geq 1$$
$$M^2 \geq 1$$

Theorem 2.4:

If C_T be quasihyponormal on the space $H^2(\beta)$, then $T(0) = 0$.

Proof :

Let C_T be quasihyponormal on $H^2(\beta)$ and k^{β}_{0} be point evaluation at 0.

Then $\| C^*_T C_T f \|_{\beta} \leq \| C^2_T f \|_{\beta}$ for all f in $H^2(\beta)$ and if $f = k^{\beta}_{0}$, we have,

$$\begin{aligned}\| C^*_T C_T k^{\beta}_{0} \|^2_{\beta} = \| C^*_T k^{\beta}_{0} \|^2_{\beta} = \| k^{\beta}_{T(0)} \|^2_{\beta} &= \sum_{n=0}^{\infty} \frac{1}{\beta^2_n} | T(0) |^{2n} \\ &\leq \| C^2_T k^{\beta}_{0} \|^2_{\beta} \\ &= \| C_T C_T k^{\beta}_{0} \|^2_{\beta} \\ &= \| C_T k^{\beta}_{0} \|^2_{\beta} \\ &= \| k^{\beta}_{0} \|^2_{\beta} \\ &= 1\end{aligned}$$

which implies, since $\beta_0 = 1$, that $T(0) = 0$.

Theorem 2.5:

If C_T is of Class A on $H^2(\beta)$ then $T(0) = 0$.

Proof :

If C_T is of Class A, implies

$$(C^*_T C_T)^2 \leq C^*_T | C_T |^2 C_T$$
$$(C^*_T C_T)^2 \leq C^*_T (C^*_T C_T) C_T$$

$$
\begin{aligned}
(C_T^{*} C_T)^2 &\le C_T^{*2}\ C_T^2 \\
\left\langle C_T^{*} C_T C_T^{*} C_T f, f \right\rangle &\le \left\langle C_T^{*2}\ C_T^2 f, f \right\rangle \qquad \text{for all f in } H^2(\beta). \\
\left\langle C_T^{*} C_T f, C_T^{*} C_T f \right\rangle &\le \left\langle C_T^2 f, C_T^2 f \right\rangle \\
\| C_T^{*} C_T f \|_\beta^2 &\le \| C_T^2\ f \|_\beta^2
\end{aligned}
$$

Let f = k_0^β, we have,

$$
\begin{aligned}
\| C_T^{*} C_T k_0^\beta \|_\beta^2 &\le \| C_T^2\ k_0^\beta \|_\beta^2 \\
\| C_T^{*}\ k_0^\beta \|_\beta^2 &\le \| C_T C_T\ k_0^\beta \|_\beta^2 \\
\| k_{T(0)}^\beta \|_\beta^2 &\le \| C_T\ k_0^\beta \|_\beta^2 = \| k_0^\beta \|_\beta^2 = 1 \\
\| k_{T(0)}^\beta \|_\beta^2 &\le 1, \text{ by theorem 4, which implies that } T(0) = 0.
\end{aligned}
$$

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