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A Computational Method for Solving Singularly Perturbed Two-Point Singular Boundary Value Problem

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Abstract

In this paper, we present a new method for solving singularly perturbed two-point singular boundary value problem. Its exact solution is represented in the form of series in reproducing kernel space. In the mean time, the n-term approximation $u_n(x)$ to the exact solution $u(x)$ is obtained and is proved to converge to the exact solution. Some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method indicate the method is simple and effective.

Keywords: Exact solution; singularly perturbed two-point singular boundary value problem; reproducing kernel

1 Introduction

In this paper, we consider the following singularly perturbed two-point singular boundary value problem in reproducing kernel space

$$\begin{cases} \varepsilon u''(x) + \frac{k}{p(x)}u'(x) + \frac{s}{q(x)}u(x) = f(x), \ \ 0 < x \leq 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$, $u(x) \in W_2^3[0,1]$, $p(x) = O(x^\alpha)(\alpha > 0)$ and $q(x) = O(x^\beta)(\beta > 0)$.
The singularly perturbed differential equations arise in a variety of differential applied mathematics, fluid mechanics, quantum mechanics, optimal control,

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gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations. Therefore, the problem has attracted much attention and has been studied by many authors. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to go for non-classical method. Some non-classical methods are suggested by various authors[1-4]. But only few authors have developed numerical methods for singularly perturbed two-point singular boundary value problem. R.K.Mohanty and his co-workers have given the numerical solution of singularly perturbed two-point singular boundary value problem using convergent tension spline method and non-uniform mesh tension spline method[5-6].

In this paper, we will give the representation of exact solution to $Eq.(1)$ and approximate solution in the reproducing kernel space under the assumption that the solution to $Eq.(1)$ is unique. The approach is simple and effective. For convenience, we take $p(x) = x$ and $q(x) = x^2$ in $Eq.(1)$. After multiplying $Eq.(1)$ by x^2, we find that

$$\begin{cases} \varepsilon x^2u''(x) + xu'(x) + u(x) = x^2 f(x), \ \ 0 \leq x \leq 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$, $u \in W_2^3[0,1], x^2 f(x) \in W_2^1[0,1]$.
Clearly, the solution of $Eq.(2)$ is the solution of $Eq.(1)$. So we only need to obtain the solution of $Eq.(2)$.
Write $F(x) = x^2 f(x)$ simply and put $Lu \equiv x^2u''(x) + xu'(x) + u(x)$. Then $Eq.(1)$ can further be converted into following form

$$\begin{cases} Lu(x) = F(x), \ \ 0 \leq x \leq 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{3}$$

where $0 < \varepsilon \ll 1$, $u \in W_2^3[0,1], F(x) \in W_2^1[0,1]$. $W_2^1[0,1]$ and $W_2^3[0,1]$ are defined in the following section.

2 Several Reproducing Kernel Spaces

1 The reproducing kernel space $W_2^3[0,1]$

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real value functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$(u(y), v(y))_{W_2^3} = \int_0^1 (36uv + 49u'v' + 14u''v'' + u^{(3)}v^{(3)})dy, \tag{4}$$

and the norm $\| u \|_{W_2^3}$ is denoted by $\| u \|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Theorem 2.1. *The space* $W_2^3[0,1]$ *is a reproducing kernel space. That is, for any* $u(y) \in W_2^3[0,1]$ *and each fixed* $x \in [0,1]$, *there exists* $R_x(y) \in W_2^3[0,1]$, $y \in [0,1]$, *such that* $(u(y), R_x(y))_{W_2^3} = u(x)$. *The reproducing kernel* $R_x(y)$ *can be denoted by*

$$R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y} + c_5e^{3y} + c_6e^{-3y}, & y \le x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y} + d_5e^{3y} + d_6e^{-3y}, & y > x. \end{cases} \tag{5}$$

The coefficients of the reproducing kernel $R_x(y)$ and the proof of Theorem (2.1) are given in [8].

2 The reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_0^1 (uv + u'v')dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,1]$. In [7], the authors had proved that $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh(|x-y|-1)].$$

3 The solution of Eq.(3)

In this section, the solution of *Eq.*(3) is given in the reproducing kernel space $W_2^3[0,1]$.

In *Eq.*(3), it is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^\infty$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1, 2, ...). \tag{6}$$

Theorem 3.1. *For Eq.*(3), *if* $\{x_i\}_{i=1}^\infty$ *is dense on* $[0,1]$, *then* $\{\psi_i(x)\}_{i=1}^\infty$ *is the complete system of* $W_2^3[0,1]$ *and* $\psi_i(x) = L_yR_x(y)|_{y=x_i}$.

Proof. We have

$$\begin{aligned} \psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), R_x(y)) \\ &= (\varphi_i(y), L_yR_x(y)) = L_yR_x(y)|_{y=x_i}. \end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y.

Clearly, $\psi_i(x) \in W_2^3[0,1]$.
For each fixed $u(x) \in W_2^3[0,1]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, ...)$, which means that,

$$(u(x), (L^*\varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0. \tag{7}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of Eq.*(3) *is unique, then the solution of Eq.*(3) *is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{8}$$

Proof. Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0,1]$.
Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}(u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(f(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \end{aligned} \tag{9}$$

So, the proof of the theorem is complete. □

Now, the approximate solution $u_n(x)$ can be obtained by the n-term intercept of the exact solution $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{10}$$

Theorem 3.3. *Assume* $u(x)$ *is the solution of Eq.*(3) *and* $r_n(x)$ *is the error between the approximate* $u_n(x)$ *and the exact solution* $u(x)$*. Then the error* $r_n(x)$ *is monotone decreasing in the sense of* $\|\cdot\|_{W_2^3}$.

Proof. From (9), (10), it follows that

$$\begin{aligned} \| r_n \|_{W_2^3} &= \| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \overline{\psi}_i(x) \|_{W_2^3} \\ &= \sum_{i=n+1}^{\infty} (\sum_{k=1}^{i} \beta_{ik} f(x_k))^2. \end{aligned} \tag{11}$$

(11) shows that the error r_n is monotone decreasing in the sense of $\| \cdot \|_{W_2^3}$. The proof is complete. □

4 Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 4.2. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1

Considering equation

$$\begin{cases} \varepsilon u''(x) + \frac{1}{x} u'(x) + \frac{1}{x^2} u(x) = f(x), 0 < x \le 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

where $f(x) = \frac{2}{x} - 2\varepsilon - 3$. The true solution is $x - x^2$. Using our method, we choose 26 points on $[0, 1]$ and take $\varepsilon = 10^{-2}, 10^{-6}, 10^{-10}$ respectively. The numerical results are given in the following table 1,2,3.

Example 2

Considering equation

$$\begin{cases} \varepsilon u''(x) + \frac{1}{x \sin x} u'(x) + \frac{1}{x^2} u(x) = f(x), 0 < x \le 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

where $f(x) = \frac{\pi \cos(\pi x) \csc x}{x} - \varepsilon \pi^2 \sin(\pi x) + \frac{\sin(\pi x)}{x^2}$. The true solution is $\sin(\pi x)$. Using our method, we choose 26 points on $[0, 1]$ and take $\varepsilon = 10^{-2}, 10^{-6}, 10^{-10}$ respectively. The numerical results are given in the following table 4,5,6.

Table 1: Numerical results for example 1($n = 26, \varepsilon = 10^{-2}$).

x	True solution u(x)	Approximate solution u_{26}	absolute error
0.0001	9.999E-05	9.99901E-05	1.2E-10
0.08	0.0736	0.0736	3.7E-11
0.16	0.1344	0.1344	8.3E-09
0.32	0.2176	0.2176	3.5E-08
0.48	0.2496	0.2496	8.0E-08
0.64	0.2304	0.2306	1.5E-07
0.80	0.1600	0.1600	2.4E-07
0.96	0.0384	0.0383996	4.4E-07
1.00	0	0	0

Table 2: Numerical results for example 1($n = 26, \varepsilon = 10^{-6}$).

x	True solution u(x)	Approximate solution u_{26}	absolute error
0.0001	9.999E-05	9.99901E-05	1.2E-10
0.08	0.0736	0.0736	1.1E-09
0.16	0.1344	0.1344	4.1E-09
0.32	0.2176	0.2176	1.4E-08
0.48	0.2496	0.2496	2.2E-08
0.64	0.2304	0.2306	2.5E-08
0.80	0.1600	0.1600	2.1E-08
0.96	0.0384	0.0384	1.2E-08
1.00	0	0	0

Table 3: Numerical results for example 1($n = 26, \varepsilon = 10^{-10}$).

x	True solution u(x)	Approximate solution u_{26}	absolute error
0.0001	9.999E-05	9.99901E-05	1.2E-10
0.08	0.0736	0.0736	1.1E-09
0.16	0.1344	0.1344	4.1E-09
0.32	0.2176	0.2176	1.4E-08
0.48	0.2496	0.2496	2.2E-08
0.64	0.2304	0.2306	2.5E-08
0.80	0.1600	0.1600	2.1E-08
0.96	0.0384	0.0384	1.2E-08
1.00	0	0	0

Table 4: Numerical results for example 2$(n = 26, \varepsilon = 10^{-2})$.

x	True solution u(x)	Approximate solution u_{25}	absolute error
0.0001	0.0314159	0.0314157	1.8E-09
0.08	0.24869	0.248643	4.6E-05
0.16	0.481754	0.481702	5.1E-05
0.32	0.844328	0.844275	5.2E-05
0.48	0.998027	0.997974	5.2E-05
0.64	0.904827	0.904776	5.1E-05
0.80	0.587785	0.587736	4.9E-05
0.96	0.125333	0.125274	5.9E-05
1.00	0	0	0

Table 5: Numerical results for example 2$(n = 26, \varepsilon = 10^{-6})$.

x	True solution u(x)	Approximate solution u_{25}	absolute error
0.0001	0.0314159	0.0314157	1.8E-09
0.08	0.24869	0.248643	4.6E-05
0.16	0.481754	0.481702	5.1E-05
0.32	0.844328	0.844275	5.2E-05
0.48	0.998027	0.997974	5.2E-05
0.64	0.904827	0.904776	5.1E-05
0.80	0.587785	0.587736	4.9E-05
0.96	0.125333	0.125274	5.9E-05
1.00	0	0	0

Table 6: Numerical results for example 2$(n = 26, \varepsilon = 10^{-10})$.

x	True solution u(x)	Approximate solution u_{25}	absolute error
0.0001	0.0314159	0.0314157	1.8E-09
0.08	0.24869	0.248643	4.6E-05
0.16	0.481754	0.481702	5.1E-05
0.32	0.844328	0.844275	5.2E-05
0.48	0.998027	0.997974	5.2E-05
0.64	0.904827	0.904776	5.1E-05
0.80	0.587785	0.587736	4.9E-05
0.96	0.125333	0.125274	5.9E-05
1.00	0	0	0

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