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Strong Convergence Theorems of Modified Mann Iteration Methods for Asymptotically Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, we introduce the iterative sequence for an asymptotically nonexpansive mapping and an asymptotically nonexpansive semigroup. Then we prove that such a sequence converges strongly to $P_{F(T)}x_0$ and $P_{\mathcal{F}}x_0$, respectively. This main theorem concern result of Takahashi, Takeuchi and Kubota [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. V.341, 2008, 276-286], and many others.

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1 Introduction

Let X be a real Banach Space, C a nonempty closed convex subset of X and $T : C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of *fixed points* of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. We know that a Hilbert space H satisfies Opial's condition [8], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We also know that H has Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. Infact, from

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2,$$

we get that a Hilbert space has the Kadec-Klee property.

Recall also that a one-parameter family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) *Lipschitian semigroup* on C (see, e. g., [12]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$,

(ii) $T(t+s)x = T(t)T(s)x,\ t, s \geq 0, x \in C$,

(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$,

(iv) there exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for each $t > 0$,

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|,\ x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called *nonexpansive* (or *a contraction semigroup*) if $L_t = 1$ for all $t > 0$, and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [2, 5, 9, 10]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [3, 10].

In 2003, Nakajo and Takahashi [7] introduced the following modification of the Mann iteration method for a nonexpansive mapping T of C into itself in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\},\\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ defined by (1.1) converges strongly to $P_{Fix(T)}(x_0)$. Moreover they introduced and studied an iteration process of a nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self mappings of a nonempty closed convex subset C of a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du,\\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},\\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n\cap Q_n}(x_0). \end{cases} \tag{1.2}$$

Recently, Kim and Xu [3] adapted the iteration (1.1) to an asymptotically nonexpansive mappings T of C into itself in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n,\\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\},\\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$. They prove that if $\alpha_n \leq a$ for all n and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{Fix(T)}(x_0)$. They also modified an iterative method (1.2) to the case of an asymptotically nonexpansive semigroup $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ of self mappings of a nonempty closed convex sunset C of a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily },\\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(u)x_n du,\\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\},\\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)\left[(\frac{1}{t_n}\int_0^{t_n} L_u du)^2 - 1\right](diamC)^2 \to 0$ as $n \to \infty$.

In 2007, Takahashi, Takeuchi and Kubota [10] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$ and nonexpansive semigroup $\Im = \{T(t) : 0 \leq t < \infty\}$ in a Hilbert space H. They prove the following theorem;

Theorem 1.1 *([10] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n,\\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\},\\ u_{n+1} = P_{C_{n+1}}x_0,\ n \in \mathbf{N}, \end{cases} \tag{1.5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.2 *([10] Theorem 4.4) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$ define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{1.6}$$

where $0 \leq \alpha_n \leq a < 1, 0 < \lambda_n < \infty$ for all $n \in \mathbf{N}$ and $\lambda_n \to \infty$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Ishikawa iteration processes for an asymptotically nonexpansive mapping by idear in (1.5). Let C be a closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N}, \end{cases} \tag{1.7}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$.

The second purpose of this paper is to study the modified Ishikawa iteration process for an asymptotically nonexpansive semigroup. Let C be a closed bounded convex subset of a Hilbert space H, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be asymptotically nonexpansive semigroup of self mappings of a nonempty closed convex sunset C of a Hilbert space such that $\mathcal{F} \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbf{N} \end{cases} \tag{1.8}$$

where $\widetilde{\theta}_n = (1-\alpha_n)\left[\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)^2 - 1\right](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbf{N}$ and $\lambda_n \to \infty$.

We shall prove that both iteration processes (1.7) and (1.8) converge strongly to a fixed point of T and a common fixed point of $\mathcal{T}$, respectively, provided the sequence $\{\alpha_n\}$ is bounded from above.

2 Preliminary

In this section, we collect some lemmas which will be used in the proof for the main result in next section.

Lemma 2.1 *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x||^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.2 *[4] Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*
(i) $x_n \rightharpoonup z$ and
(ii) $Tx_n - x_n \to 0$.
Then $z \in Fix(T)$.

Lemma 2.3 *[3] Let C be a nonempty bounded closed convex subset of a Hilbert spaces H and $\Im = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties*
a) $x_n \rightharpoonup z$; and
b) $\limsup_{t\to\infty}\limsup_{n\to\infty}\|T(t)x_n - x_n\| = 0$,
then $z \in F(\Im)$.

Lemma 2.4 *[3] Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\Im = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Then it holds that*

$$\limsup_{s\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(u)xdu - T(s)\left(\frac{1}{t}\int_0^t T(u)xdu\right)\right\| = 0.$$

3 Main Results

In this section, we prove strong convergence theorems by hybrid methods for asymptotically nonexpansive mappings in Hilbert spaces.

Theorem 3.1 *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, Then $\{x_n\}$ generated by (1.7) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbf{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_k$ for some $k \in \mathbf{N}$. Then we have, for $u \in F(T) \subset C_k$

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\
&= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\
&\le \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\
&\le \alpha_k\|x_k - u\|^2 + (1-\alpha_k)k_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (\alpha_k + (1-\alpha_k)k_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)\|x_k - u\|^2 \\
&\le \|x_k - u\|^2 + (1-\alpha_k)(k_k^2 - 1)(diamC)^2 \\
&= \|x_k - u\|^2 + \theta_k \text{ with } \theta_k \to 0.
\end{aligned}$$

It follows that $u \in C_{k+1}$ and $F(T) \subset C_{k+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbf{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbf{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbf{N}$. Let $z_m \in C_{k+1} \subset C_k$ with $z_m \to z$. Since C_k is closed, $z \in C_k$ and $\|y_k - z_m\|^2 \le \|z_m - x_k\|^2 + \theta_k$. Then

$$\begin{aligned}
\|y_k - z\|^2 &= \|y_k - z_m + z_m - z\|^2 \\
&= \|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle \\
&\le \|z_m - x_k\|^2 + \theta_k + \|z_m - z\|^2 + 2\|y_k - z_m\|\|z_m - z\|.
\end{aligned}$$

Taking $m \to \infty$,

$$\|y_k - z\|^2 \le \|z - x_k\|^2 + \theta_k.$$

Hence $z \in C_{k+1}$. Let $x, y \in C_{k+1} \subset C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$ and $\|y_k - x\|^2 \le \|x - x_k\|^2 + \theta_k, \|y_k - y\|^2 \le \|y - x_k\|^2 + \theta_k$, we have

$$\begin{aligned}
\|y_k - z\|^2 &= \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|(y_k - x) - (y_k - y)\|^2 \\
&\le \alpha(\|x - x_k\|^2 + \theta_k) + (1-\alpha)(\|y - x_k\|^2 + \theta_k) - \alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_k\|^2 + (1-\alpha)\|y - x_k\|^2 - \alpha(1-\alpha)\|(x_k - x) - (x_k - y)\|^2 + \theta_k \\
&= \|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k \\
&= \|x_k - z\|^2 + \theta_k.
\end{aligned}$$

Then $z \in C_{k+1}$, it follows that C_{k+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0, \text{ for all } y \in C_n.$$

Since $F(T) \subset C_n$, we have

$$\langle x_0 - x_n, x_n - u\rangle \ge 0 \text{ for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{1}$$

So, for $u \in F(T)$, we have

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - u\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\| \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\|x_0 - u\|$$

hence

$$\|x_0 - x_n\| \le \|x_0 - u\| \text{ for all } u \in F(T) \text{ and } n \in \mathbf{N}. \tag{2}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0 \text{ for all } n \in \mathbf{N}. \tag{3}$$

So, for $x_{n+1} \in C_n$, we have, for $n \in \mathbf{N}$

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\| \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\|\|x_0 - x_{n+1}\|$$

hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\| \text{ for all } n \in \mathbf{N}. \tag{4}$$

From (2) we have $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n, \tag{5}$$

which implies that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

Further, we have

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1 - \alpha_n)T^n x_n - x_n\| \\ &= (1 - \alpha_n)\|T^n x_n - x_n\|. \end{aligned}$$

From (5), we have

$$\begin{aligned}\|T^n x_n - x_n\| &= \tfrac{1}{(1-\alpha_n)}\|y_n - x_n\| \\ &\le \tfrac{1}{(1-a)}\|y_n - x_n\| \\ &= \tfrac{1}{(1-a)}\|y_n - x_{n+1} + x_{n+1} - x_n\| \\ &\le \tfrac{1}{(1-a)}\|y_n - x_{n+1}\| + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &\le \tfrac{1}{(1-a)}(\|x_n - x_{n+1}\| + \sqrt{\theta_n}) + \tfrac{1}{(1-a)}\|x_{n+1} - x_n\| \\ &= \tfrac{2}{(1-a)}\|x_n - x_{n+1}\| + \tfrac{1}{(1-a)}\sqrt{\theta_n}.\end{aligned}$$

Hence

$$\|T^n x_n - x_n\| \le \frac{2}{(1-a)}\|x_n - x_{n+1}\| + \frac{1}{(1-a)}\sqrt{\theta_n} \to 0.$$

Putting

$$k_\infty = \sup\{k_n : n \ge 1\} < \infty,$$

we deduce that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\le k_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+k_\infty)\|x_n - x_{n+1}\| \to 0. \quad (6)\end{aligned}$$

By (6), Lemma 2.2 and boundedness of $\{x_n\}$ we obtain $\emptyset \ne \omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \le \|z_0 - x_0\|$ for all $n \ge 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| \le \|x_0 - w\| &\le \liminf_{n\to\infty}\|x_0 - x_n\| \\ &\le \limsup_{n\to\infty}\|x_0 - x_n\| \le \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\begin{aligned}\|x_n - z_0\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2 \\ &\le 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.\end{aligned}$$

This complete the proof. ◇

Now, we present the strong convergence theorem of asymptotically nonexpansive semigroups on C in a Hilbert space.

Suppose that $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ is an asymptotically nonexpansive semigroup defined on a nonempty closed convex bounded subset C of a Hilbert space H. Recall that we use L_t^T to denote the Lipschitzian constant of the mapping $T(t)$. In the rest of this section, we put $L_\infty = \sup\{L_t^T\}$ and we use $Fix(\mathcal{T})$ to denote the fixed point set of $\mathcal{T}$. Furthermore, we use $\mathcal{F} := Fix(\mathcal{T})$ to denote the set of fixed points of asymptotically nonexpansive semigroups. Note that the boundedness of C implies that $Fix(\mathcal{T})$ is nonempty (see [11]) and we assume throughout in this theorem that the set of fixed point F is nonempty.

Theorem 3.2 *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter asymptotically nonexpansive of C into itself such that $\mathcal{F} := Fix(\mathcal{T}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$. Then $\{x_n\}$ generated by (1.8) converges strongly to $z_0 = P_{\mathcal{F}}x_0$.*

Proof. First, we observe that $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. Since $F(\Im) \subset C = C_1$. Let $F(\Im) \subset C_k$ for some $k \in \mathbf{N}$. For all $z \in F(\Im) \subset C_k$ we have

$$\begin{aligned}
\|y_k - z\|^2 &= \left\|\alpha_k x_k + (1-\alpha_k)\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&= \left\|\alpha_k(x_k - z) + (1-\alpha_k)(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z)\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left\|\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} T(s)x_k ds - z\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} \|T(s)x_k - z\| ds\right)^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s\|x_k - z\| ds\right)^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2 \|x_k - z\|^2 \\
&\leq \|x_k - z\|^2 + (1-\alpha_k)\left(\tfrac{1}{\lambda_k}\textstyle\int_0^{\lambda_k} L_s ds\right)^2 (diamC)^2 \\
&= \|x_k - z\|^2 + \widetilde{\theta}_k
\end{aligned}$$

So, $z \in C_{k+1}$. Hence $F(\Im) \subset C_n$ for all $n \in \mathbf{N}$. By the same argument as in the proof of Theorem 3.1, C_n is closed and convex, $\{x_n\}$ is well-defined. Also, similar to the proof of Theorem 3.1, we can show that

$$\|x_n - x_{n+1}\| \to 0. \tag{7}$$

We can deduce that for all $0 \leq t < \infty$,

$$\begin{aligned}
\|T(t)x_n - x_n\| &= \left\|T(t)x_n - T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right)\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&\quad + \left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\leq (L_\infty + 1)\left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&\quad + \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\| \\
&:= (L_\infty + 1)A_n + B_n(t),
\end{aligned} \tag{8}$$

where $A_n := \left\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x_n\right\|$ and

$$B_n := \left\|T(t)\left(\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right) - \tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds\right\|.$$

We claim that

(i) $\lim_{n\to\infty} A_n = 0$; and

(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} B_n(t) = 0$.

By Lemma 2.3, we have that (ii) is true, while (i) is verified by the following argument. By the definition of y_n we have

$$\begin{aligned}
A_n &= \left\|\tfrac{1}{\lambda_n}\textstyle\int_0^{\lambda_n} T(s)x_n ds - x_n\right\| \\
&= \tfrac{1}{1-\alpha_n}\|y_n - x_n\|
\end{aligned}$$

$$\begin{aligned} &\le \tfrac{1}{1-a}\|y_n - x_n\| \\ &\le \tfrac{1}{1-a}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|). \end{aligned} \tag{9}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n$$

which in turn implies that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\widetilde{\theta}_n}.$$

It follows from (9) that

$$A_n \le \frac{1}{1-a}\left(2\|x_{n+1} - x_n\| + \sqrt{\widetilde{\theta}_n}\right) \to 0.$$

We thus conclude from (8) that

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0.$$

We note that by Lemma 2.3 that every weak limit point of $\{x_n\}$ is a number of $F(\Im)$. Repeating the last of the proof of Theorem 2.2 [4], we can prove that $\omega_w(x_n) = \{P_{F(\Im)}\}$. Hence $\{x_n\}$ weakly converges to $P_{F(\Im)}$, and therefore the convergence is strong. This complete the proof. $\diamond$

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