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Some Limit Theorems for Sums of Random Sequence

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Abstract: This paper presents some limit theorems for arbitrary continuous random sequence, like many previous works, the theorems directly gives the concept logarithmic likelihood ratio, as a measure of *dissimilarity* between one distribution and the reference distributions. In particularly, we give lower and upper bounds for the deviation under $Chung-Teicher$'s type conditions.

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Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{\mathcal{F}_n, n \in \mathbb{N}\}$ be an increasing sequence of sub-$\sigma-$fields of $\mathcal{F}$, and suppose that $\{\xi_n, \mathcal{F}_n, n \in \mathbb{N}\}$ be a stochastic sequence defined on this probability space, with the joint distribution densities

$$p_n(x_1, \cdots, x_n) > 0, \quad n \in \mathbb{N} \tag{1.1}$$

and $p_k(x_k), k \in \mathbb{N}$ be the marginal density functions of them, let

$$\Pi = \pi_n(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} p_k(x_k), \quad n \in \mathbb{N}. \tag{1.2}$$

Definition 1 *Let $\{\xi_n, n \in \mathbb{N}\}$ be a sequence of random variables, set*

$$T_n(\omega) = \frac{\pi_n(\xi_1, \cdots, \xi_n)}{p_n(\xi_1, \cdots, \xi_n)}, \gamma_n(\omega) = \log T_n(\omega), \tag{1.3}$$

The random variable

$$\gamma(\omega) = -\liminf_{n} \frac{\gamma_n(\omega)}{n}. \tag{1.4}$$

is called the limit logarithmic likelihood ratio, relative to the product of one-dimensional marginals of $\{\xi_n, n \in \mathbb{N}\}$, where log *is the natural logarithm,* ω *is* *the sample point, and ξ_k stands for $\xi_k(\omega)$.*

The basis for proving the Strong Limit Theorems is the convergence of likelihood ratio which we, however, quote because of its central role in this paper.

Lemma 1*(See Chung [4])Let $p_n(x_1, \cdots, x_n), h_n(x_1, \cdots, x_n)$ be two probability density functions on $\{\Omega, \mathcal{F}, P\}$ let*

$$\Lambda_n(\omega) = \frac{h_n(\xi_1, \cdots, \xi_n)}{p_n(\xi_1, \cdots, \xi_n)}.$$

then

$$\lim_n \Lambda_n(\omega) = \Lambda_\infty(\omega) < +\infty, \quad a.s. \tag{1.5}$$

and hence

$$\limsup n^{-1} \log \Lambda_n(\omega) \le 0, \quad a.s. \tag{1.6}$$

Let $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel functions and let $\alpha_n \ge 1, \beta_n \le 2, C_n > 0, D_n > 0 (n \in \mathbb{N})$ be constants satisfying(See Cvetan et al [2] and Yang [3])

$$v \le u \Longrightarrow C_n \frac{u^{\alpha_n}}{v^{\alpha_n}} \le \frac{\varphi_n(u)}{\varphi_n(v)} \le D_n \frac{u^{\beta_n}}{v^{\beta_n}} \tag{1.7}$$

Main Results and Proofs

Theorem 1*Let $\{\xi_n, n \in \mathbb{N}\}, \{\varphi_n(x), n \in \mathbb{N}\}, \gamma(\omega)$ be defined as above and let $\{a_n, n \in \mathbb{N}\}$ be a sequences of non-zero reals. Further set*

$$\mathcal{J} = \{\omega : \gamma(\omega) < +\infty\}. \tag{2.1}$$

If

$$\limsup_n \frac{1}{n} \sum_{k=1}^n A_k E[\frac{\varphi_k(|\xi_k|)}{\varphi_k(|a_k|)}] = c < \infty, \tag{2.2}$$

where $A_n = \max(1/C_n, D_n)$, then

$$\limsup_n \frac{1}{n} \sum_{k=1}^n \frac{\xi_k - E\xi_k}{a_k} \le \tau(\gamma(\omega), c) + c, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.3}$$

where

$$\tau(x, y) = \inf\{\psi(\lambda, x, y), \lambda > 0\}, \quad 0 \le x, y < +\infty. \tag{2.4}$$

$$\psi(\lambda, x, y) = \frac{x}{\lambda} + \frac{1}{2}\lambda e^{2|\lambda|} y, \quad 0 \le x, y < +\infty, \lambda > 0. \tag{2.5}$$

and

$$\tau(x, y) \ge 0, \tau(0, y) = \tau(x, 0) = 0, 0 \le x, y < +\infty. \tag{2.6}$$

$$\lim_{x \to 0+} \tau(x, y) = 0. \tag{2.7}$$

$I_{[\cdot]}$ denotes the indicator function.

Proof. Let $\xi'_n = \xi_n I_{[|\xi_n|\le |a_n|]}, n \in \mathbb{N}$. It follows from (1.7) that on the set $\{x : |x| \ge |a_n|\}$, we have

$$\frac{|x|}{|a_n|} \le \frac{|x|^{\alpha_n}}{|a_n|^{\alpha_n}} \le A_n \frac{\varphi_n(|x|)}{\varphi_n(|a_n|)},$$

Thus we have

$$\sum_{n=1}^{\infty} P(\xi_n \ne \xi'_n) = \sum_{n=1}^{\infty} P(|\xi_n| > |a_n|) = \sum_{n=1}^{\infty} EI_{[|\xi_n|\ge |a_n|]}$$

$$\le \sum_{n=1}^{\infty} E[\frac{|\xi_n|}{|a_n|}] \le \sum_{n=1}^{\infty} A_n \frac{E[\varphi_n(|\xi_n|)]}{\varphi_n(|a_n|)} < \infty$$

which implies

$$P(\xi_n \ne \xi'_n, i.o.) = 0$$

and hence

$$\sum_{n=1}^{\infty} \frac{\xi_n - \xi'_n}{a_n} \quad converges \quad a.s. \tag{2.8}$$

Since

$$\frac{E\xi'_n - E\xi_n}{a_n} \le \frac{E|\xi'_n - \xi_n|}{|a_n|} = E\{\frac{|\xi_n|}{|a_n|} I_{[|\xi_n| > |a_n|]}\}$$

$$\le A_n E[\frac{\varphi_n(|\xi_n|)}{\varphi_n(|a_n|)} I_{[|\xi_n|\ge |a_n|]}] \le A_n E[\frac{\varphi_n(|\xi_n|)}{\varphi_n(|a_n|)}]$$

By (2.2), we have

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} \frac{E\xi'_k - E\xi_k}{a_k} \le c \quad a.s. \tag{2.9}$$

Put $\eta_n = (\xi'_n - E\xi'_n)/a_n, n \in \mathbb{N}$, then $|\eta_n| \le 2$, for $\lambda \in \mathbb{R}$, let us set

$$Q_k(\lambda) = E[\exp(\lambda \eta_k)] = \int_{|x_k|\le |a_k|} p_k(x_k) \exp[\frac{\lambda(x_k - E\xi'_k)}{a_k}] dx_k.$$

$$p(\lambda; x_k) = \frac{p_k(x_k) \exp\{\lambda[x_k I_{[|x_k|\le |a_k|]} - E\xi'_k]/a_k\}}{Q_k(\lambda)}; \;\; q_n(\lambda; x_1, \cdots, x_n) = \prod_{k=1}^{n} p_k(\lambda; x_k).$$

We define the likelihood ratio as below

$$\Lambda_n(\lambda; \omega) = \frac{q_n(\lambda; \xi_1, \cdots, \xi_n)}{p_n(\xi_1, \cdots, \xi_n)}.$$

The lemma can be rewrite as

$$\limsup_n n^{-1} \log \Lambda_n(\lambda; \omega) \le 0, \quad a.s.$$

Thus we have

$$\limsup_n \frac{1}{n} [\lambda \sum_{k=1}^{n} \eta_k + \gamma_n(\omega) - \sum_{k=1}^{n} \log Q_k(\lambda)] \le 0, \; a.s. \tag{2.10}$$

(1.4) and (2.10) imply

$$\limsup_n \frac{\lambda}{n}\sum_{k=1}^{n}\eta_k \leq \gamma(\omega)+\limsup_n \frac{1}{n}\sum_{k=1}^{n}\log Q_k(\lambda), \quad a.s.\ \omega\in\mathcal{J}$$

From the inequality $0\leq e^x-1-x\leq \frac{1}{2}x^2e^{|x|}\quad for\ all\ \ x\in R$, we have

$$\begin{aligned} 0\leq Q_k(\lambda)-1 &= E[\exp(\lambda\eta_k)-1-\lambda\eta_k] \\ &\leq \frac{1}{2}\lambda^2e^{2|\lambda|}E\eta_k^2\leq \frac{1}{2}\lambda^2e^{2|\lambda|}E[\frac{\xi_k^{'}}{a_k}]^2. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned} E[\frac{\xi_k^{'}}{a_k}]^2 &= \int_{|x|\leq|a_k|}\frac{x^2}{a_k^2}dF_k(x)\leq\int_{|x|\leq|a_k|}[\frac{|x|}{|a_k|}]^{\beta_k}dF_k(x) \\ &\leq \int_{|x|\leq|a_k|}A_k\frac{\varphi_k(|x|)}{\varphi_k(|a_k|)}dF_k(x)\leq A_kE[\frac{\varphi_k(|\xi_k|)}{\varphi_k(|a_k|)}]. \end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have

$$0\leq Q_k(\lambda)-1\leq\frac{1}{2}\lambda^2e^{2|\lambda|}A_kE\frac{\varphi_k(|\xi_k|)}{\varphi_k(|a_k|)}.$$

Clearly

$$\begin{aligned} 0 &\leq \limsup_n\frac{1}{n}\sum_{k=1}^{n}[Q_k(\lambda)-1] \\ &\leq \frac{1}{2}\lambda^2e^{2|\lambda|}\limsup_n\frac{1}{n}\sum_{k=1}^{n}A_kE[\frac{\varphi_k(|\xi_k|)}{\varphi_k(|a_k|)}]\leq\frac{1}{2}\lambda^2e^{2|\lambda|}c\quad a.s. \end{aligned} \tag{2.13}$$

(2.13) together with the inequality $0\leq\log x\leq x-1\ \ (x\geq 1)$ yield

$$0\leq\limsup_n n^{-1}\sum_{k=1}^{n}\log Q_k(\lambda)\leq\frac{1}{2}\lambda^2e^{2|\lambda|}c,\ \ \omega\in\mathcal{J}.$$

Hence we have

$$\limsup_n\frac{\lambda}{n}\sum_{k=1}^{n}\eta_k\leq\gamma(\omega)+\frac{1}{2}\lambda^2e^{2|\lambda|}c,\ \omega\in\mathcal{J}. \tag{2.14}$$

Let $\lambda>0$, we have by (2.14)

$$\limsup_n\frac{1}{n}\sum_{k=1}^{n}\eta_k\leq\frac{\gamma(\omega)}{\lambda}+\frac{1}{2}\lambda e^{2|\lambda|}c,\ \omega\in\mathcal{J}.$$

Therefore, we have

$$\limsup_n\frac{1}{n}\sum_{k=1}^{n}\eta_k\leq\tau(\gamma(\omega),c),\quad a.s.\ \omega\in\mathcal{J} \tag{2.15}$$

Noting that

$$\frac{\xi_k - E\xi_k}{a_k} = \frac{(\xi_k - \xi'_k) + (\xi'_k - E\xi'_k) + (E\xi'_k - E\xi_k)}{a_k}$$

(2.3) follows immediately from (2.8),(2.9) and (2.15) □

Theorem 2 *Under the conditions of Theorem 1, we have*

$$\liminf_n \frac{1}{n} \sum_{k=1}^{n} \frac{\xi_k - E\xi_k}{a_k} \geq \kappa(\gamma(\omega), c) - c, \quad a.s.\ \omega \in \mathcal{J}. \tag{2.16}$$

where

$$\kappa(x,y) = \sup\{\psi(\lambda, x, y), \lambda < 0\}, \quad 0 \leq x, y < +\infty. \tag{2.17}$$

$$\psi(\lambda, x, y) = \frac{x}{\lambda} + \frac{1}{2}\lambda e^{2|\lambda|} y, \quad 0 \leq x, y < +\infty, \lambda < 0. \tag{2.18}$$

and

$$\kappa(x,y) \leq 0, \kappa(0,y) = \kappa(x,0) = 0, 0 \leq x, y < +\infty. \tag{2.19}$$

$$\lim_{x \to 0+} \kappa(x,y) = 0. \tag{2.20}$$

Corollary 1 *Under the conditions of Theorem 1, if $c = 0$ or $\gamma(\omega) = 0, a.s.$, then*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \frac{\xi_k - E\xi_k}{a_k} = 0, \quad \omega \in \mathcal{J}. \tag{2.21}$$

Proof. Since $\kappa(x,0) = \tau(x,0) = 0$ and $\kappa(0,y) = \tau(0,y) = 0$, hence (2.21) follows from (2.3) and (2.16). □

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