

Some Fixed Point Theorems in Pseudo Compact Tichonov Spaces

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Abstract

In this paper we prove some coincide point theorem in pseudo compact Tichonov spaces with two continuous mappings and one fixed point theorem in this space which generalize many results.

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Introduction and Preliminaries

In 1961 Edelstein [2] established the existence of a unique fixed point of a self map T of a compact metric space satisfying the Banach inequality which is the generalization of Banach fixed point theorem. Jain and Dixit [3], Pathak [6], Khan and Sharma [4] find some valuable results in pseudo compact tichonov spaces. More recently Namdeo and Fisher [5], Popa and Telci [7], Sahu [8] find some valuable results in compact metric space.

Definition 1.1 Fixed point space: Let F be a self mapping. A space X is called a fixed point space if every continuous mapping F of X into itself, has a fixed point, in the sense that $F(x_0) = x_0$

Definition 1.2 A class $\{G_i\}$ of open subset of X is said to be an open cover of X , if each point in X belongs to one G_i that is $\cup_i (G_i) = X$.

A subclass of an open cover which is at least an open cover is called a subcover. A compact space is that space in which every open cover is called a subcover.

Definition 1.3 Pseudo-compact tichonov spaces: A Topological space X is said to be pseudo-compact space, if every real valued continuous function on X is bounded. It may be noted that every compact space is pseudo compact but converse is not necessarily true. However, in a metric space notation ‘compact’ and ‘pseudo compact’ coincide .By Tichonov space we mean a completely regular Hausdroff space.

Main Result

Theorem 2.1 Let X be a pseudo compact Tichnov space and μ be a non negative real valued continuous function over $X \times X$ satisfying:

$$[A] \mu(x,x) = 0 \text{ and } \mu(x,y) = \mu(x,z) + \mu(z,y) \quad \forall x,y,z \in X$$

F and G be two continuous mappings on X satisfying:

$$[B] \mu(Fx, Gy) < \max$$

$$\left[\mu(x, y), \frac{\mu(x, Fx)\mu(y, Fx) + \mu(x, Gy)\mu(y, Gy)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)}, \right. \\ \left. \frac{\mu(x, Fx)\mu(y, Gy) + \mu(x, Gy)\mu(y, Fx)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)}, \frac{\mu(x, Fx)\mu(x, Gy) + \mu(y, Fx)\mu(y, Gy)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)} \right]$$

Then F and G have coincide point in X .

Proof: Let us construct a function $T : X \rightarrow R$ as $T(x) = \mu(Fx, Gx)$ for all x in X . As F and g are continuous therefore T is also. By the compactness of X there exist u in X such that

$$Tu = \inf \{ Tx : x \in X \}$$

$$\mu(Fx, Gy) < \max$$

$$\left[\mu(u, u), \frac{\mu(u, Fu)\mu(u, Fu) + \mu(u, Gu)\mu(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}, \frac{\mu(u, Fu)\mu(u, Gu) + \mu(u, Gu)\mu(u, Fu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}, \right.$$

$$\left. \frac{\mu(u, Fu)\mu(u, Gu) + \mu(u, Gu)\mu(u, Fu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}} \right]$$

$$\mu(Fx, Gy) < \max$$

$$\left[0, \frac{\mu(u, Fu) + \mu(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}, \frac{2\mu(u, Fu)\mu(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}, \frac{2\mu(u, Fu)\mu(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}} \right]$$

There arise two cases:

Case 1. $\mu(Fu, Gu) < \frac{\mu^2(u, Fu) + \mu^2(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}$

$$\mu(Fu, Gu) < \frac{\mu^2(u, Fu) + \mu^2(u, Gu)}{2\mu(Fu, Gu)}$$

$$\mu(Fu, Gu) < \frac{\mu^2(u, Fu) + \mu^2(u, Gu)}{\mu(Fu, Gu)}$$

$$\mu^2(Fu, Gu) < \mu^2(u, Fu) + \mu^2(u, Gu)$$

$$[\mu(u, Fu) + \mu(u, Gu)]^2 = \mu^2(Fu, Gu) < \mu^2(u, Fu) + \mu^2(u, Gu)$$

Which is contradiction.

Case2: $\mu(Fu, Gu) < \frac{2\mu(u, Fu)\mu(u, Gu)}{2\{\mu(u, Fu) + \mu(u, Gu)\}}$

$$[\mu(u, Fu) + \mu(u, Gu)]^2 = \mu^2(Fu, Gu) < \mu(u, Fu)\mu(u, Gu)$$

Which again contradiction.

Therefore by case 1 and 2

$\mu(Fu, Gu) = 0$. Thus u is coincide point of F and G .

Theorem 2.2 Let X be a pseudo compact Tichnov space and μ be a non negative real valued continuous function over $X \times X$ satisfying [A]. F and G be two continuous mappings on X satisfying:

$$[B] \quad \mu(Fx, Gy) < \max$$

$$\left[\mu(x, y), \frac{\mu(x, Fx)\mu(y, Fx) + \mu(x, Gy)\mu(y, Gy)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)}, \frac{\mu(x, Fx)\mu(y, Gy) + \mu(x, Gy)\mu(y, Fx)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)}, \right. \\ \left. \frac{\mu(x, Fx)\mu(x, Gy) + \mu(y, Fx)\mu(y, Gy)}{\mu(x, Fx) + \mu(y, Fx) + \mu(x, Gy) + \mu(y, Gy)} \right]$$

$$[C] \quad FG = GF$$

Then F and G have unique common point in X .

Proof: Let us construct a function $T : X \rightarrow R$ as $T(x) = \mu(FGx, Gx)$ for all x in X . As F and g are continuous therefore T is also. By the compactness of X there exist u in X such that $Tu = \inf \{ Tx : x \in X \}$. Now we affirm that u is fixed point of F .

$$T(Fu) = \mu(FGFu, GFu) = \mu(FGFu, GFu) < \max$$

$$\left[\mu(GFu, Fu), \frac{\mu(GFu, FGFu)\mu(Fu, FGFu) + \mu(GFu, GFu)\mu(Fu, GFu)}{\mu(GFu, FGFu) + \mu(Fu, FGFu) + \mu(GFu, GFu) + \mu(Fu, GFu)}, \right. \\ \frac{\mu(GFu, FGFu)\mu(Fu, GFu) + \mu(GFu, GFu)\mu(Fu, FGFu)}{\mu(GFu, FGFu) + \mu(Fu, FGFu) + \mu(GFu, GFu) + \mu(Fu, GFu)}, \\ \left. \frac{\mu(GFu, FGFu)\mu(GFu, GFu) + \mu(Fu, FGFu)\mu(Fu, GFu)}{\mu(GFu, FGFu) + \mu(Fu, FGFu) + \mu(GFu, GFu) + \mu(Fu, GFu)} \right]$$

$$\mu(FGFu, GFu) < \max \left[\mu(GFu, Fu), \frac{\mu(GFu, FGFu)\mu(Fu, FGFu)}{2\mu(GFu, FGFu)}, \frac{\mu(GFu, FGFu)\mu(Fu, FGFu)}{2\mu(GFu, FGFu)}, \frac{\mu(Fu, FGFu)\mu(Fu, GFu)}{2\mu(GFu, Fu)} \right]$$

$$\mu(FGFu, GFu) < \max \left[\mu(GFu, Fu), \frac{\mu(Fu, FGFu)}{2}, \frac{\mu(Fu, FGFu)}{2}, \frac{\mu(Fu, FGFu)}{2} \right]$$

$$\mu(FGFu, GFu) < \max [\mu(GFu, Fu), \mu(Fu, FGFu), \mu(Fu, FGFu), \mu(Fu, FGFu)]$$

T(Fu) < T(u), which contradiction. Therefore Fu = u.

Now, since FG = GF, FG u = GF u = Gu.

Let Gu ≠ u, then μ(Gu, u) = μ(FGu, Fu) and

$$\begin{aligned} & \mu(FGu, Fu) < \max \\ & \left[\mu(u, u), \frac{\mu(u, Fu)\mu(u, Fu) + \mu(u, Gu)\mu(u, Gu)}{\mu(u, Fu) + \mu(u, Fu) + \mu(u, Gu) + \mu(u, Gu)}, \right. \\ & \left. \frac{\mu(u, Fu)\mu(u, Gu) + \mu(u, Gu)\mu(u, Fu)}{\mu(u, Fu) + \mu(u, Fu) + \mu(u, Gu) + \mu(u, Gu)}, \frac{\mu(u, Fu)\mu(u, Gu) + \mu(u, Fu)\mu(u, Gu)}{\mu(u, Fu) + \mu(u, Fu) + \mu(u, Gu) + \mu(u, Gu)} \right] \\ & < \max \left[0, \frac{\mu^2(u, Gu)}{2\mu(u, Gu)}, 0, 0 \right] \end{aligned}$$

μ(Gu, u) < μ(Gu, u). This is contradiction, therefore Gu = u.

Hence F and G have a common fixed point u. By [B] easily prove uniqueness of u.

Theorem 2.2 Let F be a continuous mapping of a compact metric space X into itself satisfying

$$\begin{aligned} [D] \quad & \mu(Fx, Fy) < \max \\ & \left[d(x, y), d(x, Fx), d(y, Fx), \frac{d(x, Fx)d(y, Fx) + d(x, Fy)d(y, Fy)}{d(x, Fx) + d(y, Fx) + d(x, Fy) + d(y, Fy)}, \right. \\ & \left. \frac{d(x, Fx)d(y, Fy) + d(x, Fy)d(y, Fx)}{d(x, Fx) + d(y, Fx) + d(x, Fy) + d(y, Fy)}, \frac{d(x, Fx)d(x, Fy) + d(y, Fx)d(y, Fy)}{d(x, Fx) + d(y, Fx) + d(x, Fy) + d(y, Fy)} \right] \end{aligned}$$

Has unique fixed point in X.

Proof: Let us construct a function T on X as Tx = d(x, Fx) for all x in X. Since F and metric D are continuous therefore T is also. By compactness of X there exists a point u in X such that Tu = inf { Tx: x ∈ X }.....(1)

Now we prove u is fixed point.

If $Tu \neq 0$, then $Fu \neq u$

Now $T(Fu) = d(Fu, F(Fu)) < \max$

$$\left[d(u, Fu), d(u, Fu), d(Fu, Fu), \frac{d(u, Fu)d(Fu, Fu) + d(u, FFu)d(Fu, FFu)}{d(u, Fu) + d(Fu, Fu) + d(u, FFu) + d(Fu, FFu)}, \right.$$

$$\left. \frac{d(u, Fu)d(Fu, FFu) + d(u, FFu)d(Fu, Fu)}{d(u, Fu) + d(Fu, Fu) + d(u, FFu) + d(Fu, FFu)}, \frac{d(u, Fu)d(u, FFu) + d(Fu, Fu)d(Fu, FFu)}{d(u, Fu) + d(Fu, Fu) + d(u, FFu) + d(Fu, FFu)} \right]$$

$$d(Fu, F(Fu)) < \max \left[d(u, Fu), d(u, Fu), 0, \frac{d(Fu, FFu)}{2}, \frac{d(Fu, FFu)}{2}, d(u, Fu) \right]$$

If $d(Fu, F(Fu)) < d(Fu, FFu)$], which is contradiction.

Therefore $d(Fu, F(Fu)) < d(u, Fu)$]

$\Rightarrow T(Fu) < T(u)$, which again contradiction to (1). Hence $u = Fu$, i.e. u is fixed point of F .

By [D] one can easily prove the uniqueness of u .

Remark: Theorem 2.3 generalize the result of Edelstein[2] and Ramakant et. al.[1]

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