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Projection Regularization Method for Convex Feasibility Problems in Banach Spaces

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Abstract

Without demanding the weak sequential continuous property for a normalized duality mapping of an uniformly smooth and uniformly convex Banach space E, we present a finite dimensional variant of the Tikhonov regularization method for finding a point in the nonempty intersection $\cap_{i=1}^N C_i$, where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set of a nonexpansive mapping $T_i : E \to E$.

Mathematics Subject Classification: 47H17

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1. Introduction

Let E be a real uniformly smooth and uniformly convex Banach space and its dual space E^* be strictly convex. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$.

We are concerned with the following convex feasibility problem:

$$\text{finding an } x_* \in C := \cap_{i=1}^N C_i, \tag{1.1}$$

where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set $F(T_i)$ of a nonexpansive mapping $T_i : E \to E, i = 1, 2, ..., N$, i.e.,

$$\|T_i x - T_i y\| \leq \|x - y\| \quad \forall x, y \in E, i = 1, ..., N.$$

Let K be a nonempty closed convex subset of E. Then for each $x \in K$, the set $I_K(x)$ denoted by

$$I_K(x) = \{y \in E : y + \lambda(z - x), z \in K, \lambda \geq 0\}$$

is called a *inward set*. A mapping $S : E \to K$ is said to satisfy the weakly inward condition, if $Sx \in \overline{I_K(x)}$ (the closure of $I_K(x)$) for each $x \in K$.

It is wellknown in [8] the iteration method

$$x_{n+1} = P(\alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T_{n+1} x_n), \tag{1.2}$$

where $x_0 \in E$ is any given initial data, $f(x) : K \to K$ is a given contractive mapping, $T_n = T_{n(\text{mod})N}$, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and P is a sunny nonexpansive retraction of E onto K.

Theorem 1.1 *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let K be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E and P is a sunny nonexpansive retaction from E onto K. Let $f : K \to K$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$, and let $T_i : E \to E, i = 1, 2, ..., N$, be nonexpansive mappings satisfying the conditions:*

(i) $\cap_{i=1}^N (F(T_i) \cap K) \neq \emptyset$;

(ii) $\cap_{i=1}^N F(T_i) = F(T_1 T_N ... T_2) = ... = F(T_N T_{N-1} ... T_1) = F(S)$, *where* $S = T_N T_{N-1} ... T_1$;

(iii) The mapping $S : K \to E$ satisfies the weakly inward condition;

For any $x_0 \in K$, let $\{x_n\}$ be the sequence defined by (1.2). If the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{i=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{i=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$,

then the sequence $\{x_n\}$ converges strongly to a point $p \in \cap_{i=1}^N (F(T_i) \cap K)$ which is the unique solution of the following variational inequality:

$$\langle p - f(p), j(p - u)\rangle \leq 0, \quad \forall u \in \cap_{i=1}^N (F(T_i) \cap K).$$

If E is a Hilbert space and $f(x) = u$ (a given point in K), then (1.2) is equivalent to

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n$$

which was introduced and studied by Bauschke [3] in 1996.

If $N = 1$, E either is a uniformly smooth Banach space or a reflexive Banach space with a weakly sequentially continuous duality mapping and K is a nonempty closed convex subset of E, $T : K \to K$ is a nonexpansive mapping, and $f : K \to K$ is a contractive mapping, then (1.2) is equivalent to the following sequence:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T x_n$$

which is studied by Xu in [13]. This algorithm is an extention of the first introduced and studied by Moudafi [9] in the setting of Hilbert space.

In this paper, we present a new approach to solve (1.1) without demanding conditions (i)-(iii) and the weak sequential continuous property of the normalized duality mapping j of E in theorem 1.1. More precisely, we present the Tikhonov regularization method in infinite dimensional Banach space E and in connection with its finite dimensional approximations for solving (1.1).

Later, the symbols $\to$ and $\rightharpoonup$ denote the strong and the weak convergence, respectively.

2. Main results

We formulate the following facts which are necessary in the proof of our results.

Denote by I the identity operator in E.

Lemma 2.1 [2] *If $A = I - T$ with a nonexpansive mapping T, then for all $x, y \in D(A)$, the domain of A,*

$$\langle A(x) - A(y), j(x - y) \rangle \geq L^{-1} R^2 \delta_E\left(\frac{\|A(x) - A(y)\|}{4R}\right), 1 < L < 1.7,$$

where $R \geq \|x\|, \|y\|$, $\delta_E(\varepsilon)$ is the modulus of convexity of the space E, and j is the normalized duality mapping of E, i.e., a mapping from E onto E^* with the following property

$$\langle x, j(x) \rangle = \|x\| \|j(x)\| = \|x\|^2 \quad \forall x \in E.$$

If E^* is strictly convex then j is single-valued. In connection with j we assume that the relation

$$\|j(x) - j(y)\| \leq c(r) \|x - y\|^\nu, 0 < \nu \leq 1,$$

where $c(r), r > 0$, is a positive increasing function on $r = \max\{\|x\|, \|y\|\}$ (see [11]) holds.

The modulus of smoothness of E is the function

$$\rho_E : [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\right\}.$$

E is uniformly smooth if and only if $\lim_{\tau\to 0}(\rho_E(\tau)/\tau) = 0$. Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, $W^p_m, 1 < p < \infty$, are uniformly smooth.

The modulus of convexity of E is the function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| : \|x\| = 1, \|y\| = 1, \|x - y\| = \varepsilon\}.$$

E is uniformly convex if and only if

$$\delta_E(\varepsilon) > 0 \quad \forall\varepsilon > 0,$$

and E is strictly convex if and only if the normalized duality mapping j of E is strictly monotone, i.e.,

$$\langle x - y, j(x) - j(y)\rangle \geq 0$$

and the symbol '=' is achieved if and only if $x = y$.

It is wellknown [1] that when E is the space of type L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, $W^p_m, 1 < p < \infty$, for $0 < \varepsilon \leq 2$ we have

$$\begin{aligned}\delta_E(\varepsilon) &\geq 16^{-1}(p-1)\varepsilon^2, \text{for} \quad 1 < p \leq 2,\\ \delta_E(\varepsilon) &\geq p^{-1}(p-1)(\varepsilon/2)^p, \text{for} \quad p \geq 2.\end{aligned}$$

Cleraly, if T is nonexpansive, then A is accretive and Lipschitz continuous with the Lipschitz constant $L_A = 2$.

Consider an operator version of the Tikhonov regularization method in the form

$$\sum_{i=1}^{N} A_i(x) + \alpha_n x = 0, A_i = I - T_i, \tag{2.1}$$

depending on the positive regularization parameter α_n that tends to zero as $n \to +\infty$. Equation (2.1) is a simple form of the following equation

$$\sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x) + \alpha_n x = 0, 0 \leq \mu_i < 1.$$

If $\mu_i = 0, i = 1, 2, ..., N$, then this algorithm is studied in [7] for finding a common fixed point of a finite family of strictly pseudocontractive mappings in Banach spaces. If $\mu_1 = 0, \mu_i < \mu_{i+1}, i = 1, 2, ..., N - 1$, then it is investigated in [5] and [6] for finding a common solution of potential monotone hemi-continuous mapping $A_i : E \to E^*$ and a common fixed point of a finite family of strictly pseudocontractive mappings in Hilbert spaces, respectively.

We have the following result.

Theorem 2.1 *(i) For each $\alpha_n > 0$, problem (2.1) has a unique solution x_n.*

(ii) If the sequence $\{\alpha_n\}$ is chosen such that

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0,$$

for any positive integer p, then

$$\lim_{n\to+\infty} x_n = x_* \in C.$$

Proof.(i) Since for each fixed $\alpha_n > 0$ the mapping $\sum_{i=1}^N A_i$ is Lipschitz continuous and accretive, then it is m-accretive [4]. Hence, equation (2.1) has a unique solution denoted by x_n for each $\alpha_n > 0$.
(ii) From (2.1) it follows

$$\sum_{i=1}^N \langle A_i(x_n), j(x_n - y)\rangle + \alpha_n\langle x_n, j(x_n - y)\rangle = 0 \quad \forall y \in C. \tag{2.2}$$

Because of $A_i(y) = 0, i = 1, ..., N$, we have

$$\sum_{i=1}^N A_i(y) = 0.$$

The last equality, (2.2) and the accretive property of A_i give

$$\langle x_n, j(x_n - y)\rangle \leq 0$$

or

$$\langle x_n - y, j(x_n - y)\rangle \leq \langle -y, j(x_n - y)\rangle \quad \forall y \in C.$$

Consequently,

$$\|x_n - y\| \leq \|y\| \quad \text{and} \quad \|x_n\| \leq 2\|y\|, y \in C. \tag{2.3}$$

Let x_{n+p} be the solution of (2.1) when α_n is replaced by α_{n+p}. Then,

$$\sum_{i=1}^N \langle A_i(x_n) - A_i(x_{n+p}), j(x_n - x_{n+p})\rangle + \alpha_n\langle x_n, j(x_n - x_{n+p})\rangle \\ - \alpha_{n+p}\langle x_{n+p}, j(x_n - x_{n+p})\rangle = 0.$$

Hence,

$$\|x_n - x_{n+p}\| \leq \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} 2\|y\|.$$

Clearly, from

$$\lim_{n\to+\infty} \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0$$

it follows that $\{x_n\}$ is a Cauchy sequence in the Banach space E. Therefore,

$$\lim_{n\to+\infty} x_n = x_* \in E.$$

Now, we shall prove that $x_* \in F(T_l), l = 1, ..., N$. For any $y \in C$ from lemma 2.1, (2.1), (2.3) and the accretive property of A_i it implies that

$$\begin{aligned}\delta_E\left(\frac{\|A_l(x_n)\|}{8\|y\|}\right) &\le L(2\|y\|)^{-2}\langle A_l(x_n), j(x_n - y)\rangle \\ &\le L(2\|y\|)^{-2}\sum_{i=1}^{N}\langle A_i(x_n), j(x_n - y)\rangle \\ &\le L(2\|y\|)^{-2}\alpha_n^{1-\mu_l}\langle -x_n, j(x_n - y)\rangle \\ &\le L(2\|y\|)^{-2}\alpha_n^{1-\mu_l}\langle -y, j(x_n - y)\rangle \le \frac{L}{4}\alpha_n.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|A_l(x_n)\| = 0.$$

Since A_l is continuous, then $A_l(x_*) = 0$, i.e., $x_* \in F(T_l)$. It means that $x_* \in C$. Theorem is proved.

Now, consider the finite dimensional approximations of (2.1) in the form

$$\sum_{i=1}^{N} A_i^n(z) + \alpha_n z = 0, z \in E_n, \tag{2.4}$$

where $A_i^n = P_n A_i P^n, P_n$ is a linear projection of E into its subspace E_n such that

$$E_n \subset E_{n+1}, \quad P_n x \to x, n \to +\infty, \forall x \in E.$$

Without loss of generality, assume that $\|P_n\| = 1$ [12].

Set

$$\gamma_n(y) = \|(I - P_n)y\|$$

where $y \in C$.

Theorem 2.2 *(i) For each $\alpha_n > 0$, problem (2.4) has a unique solution z_n.*
(ii) If $\gamma_n(y) = o(\alpha_n)$ for each $y \in C$, $T_i, i = 1, ..., N$ are Fréchet differentiable with

$$\|T_i'(x) - T_i'(y)\| \le L_i\|x - y\|, L_i > 0, \tag{2.5}$$

and the sequence $\{\alpha_n\}$ is chosen such that

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0,$$

for any positive integer p, then

$$\lim_{n\to+\infty} z_n = z_* \in C.$$

Proof.(i) By the similar argument as in the proof for (i) in theorem 2.1, we can conclude that equation (2.5) has a unique solution denoted by z_n for each $\alpha_n > 0$.
(ii) From (2.4) and the property of j^n, the normalized duality mapping of E_n, it follows

$$\alpha_n\|z_n - y_n\|^2 = -\alpha_n\langle y_n, j^n(z_n - y_n\rangle + \sum_{i=1}^{N}\langle -A_i^n(z_n), j^n(z_n - y_n)\rangle$$
$$\leq \alpha_n\langle y_n, j^n(y_n - z_n)\rangle + \sum_{i=1}^{N}\langle A_i(y) - A_i(y_n), j^n(z_n - y_n)\rangle,$$

where $y_n = P_n y, y \in C$.

Since

$$T_i(y_n) = T_i(y) + T_i'(y)(P_n y - y) + r_i^n, \|r_i^n\| \leq \frac{L_i}{2}\gamma_n^2(y), i = 1, ..., N,$$

then

$$\|z_n - y_n\| \leq \|y_n\| + \gamma_n(y)\left[\sum_{i=1}^{N} 1 + \|T_i'(y)\| + \frac{L_i}{2}\gamma_n(y)\right]/\alpha_n.$$

Consequently, there exists a positive constant R such that $\|z_n\| \leq R$ for $n \geq 1$.

Let z_{n+p} be the solution of (2.4) when α_n is replaced by α_{n+p}. Note that $z_n \in E_{n+p}$ for any $p \geq 1$. Therefore, $A_i^{n+p}(z_n) = A_i^n(z_n)$. Then,

$$\sum_{i=1}^{N}\langle A_i^{n+p}(z_n) - A_i^{n+p}(z_{n+p}), j^{n+p}(z_n - z_{n+p})\rangle + \alpha_n\langle z_n, j^{n+p}(z_n - z_{n+p})\rangle$$
$$- \alpha_{n+p}\langle z_{n+p}, j^{n+p}(z_n - z_{n+p})\rangle = 0.$$

Hence,

$$\|z_n - z_{n+p}\| \leq \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n}R.$$

Clearly, from

$$\lim_{n\to+\infty}\frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0$$

it follows that $\{z_n\}$ is a Cauchy sequence in the Banach space E. Therefore,

$$\lim_{n\to+\infty} z_n = z_* \in E.$$

Now, we shall prove that $z_* \in F(T_l), l = 1, ..., N$. For any $y \in C$ from lemma 2.1, (2.4), (2.5), $P_n^2 = P_n, j^n(u) = j(u)$ for any $u \in E_n$ [10] and the accretive property of A_i it implies that

$$L^{-1}R^2\delta_E\left(\frac{\|A_l(z_n)\|}{4R}\right) \leq \langle A_l(z_n), j(z_n - y)\rangle$$
$$\leq \sum_{i=1}^{N}\langle A_i(z_n), j(z_n - y)\rangle$$

$$\leq \sum_{i=1}^{N}\langle A_i(z_n), j(z_n - y_n)\rangle + \sum_{i=1}^{N}\langle A_i(z_n), j(z_n - y) - j(z_n - y_n)\rangle$$
$$\leq \sum_{i=1}^{N}\langle A_i^n(z_n), j^n(z_n - y_n)\rangle + \sum_{i=1}^{N}\langle A_i(z_n), j(z_n - y) - j(z_n - y_n)\rangle$$
$$\leq -\alpha_n\langle y_n, j^n(z_n - y_n)\rangle + 2N\|z_n - y\|\gamma_n^{\nu}(y)$$
$$\leq \alpha_n\|y\|\|z_n - y_n\| + 2N\|z_n - y\|\gamma_n^{\nu}(y).$$

Therefore,

$$\lim_{n\to+\infty}\|A_l(z_n)\| = 0.$$

Since A_l is continuous, then $A_l(z_*) = 0$, i.e., $z_* \in F(T_l)$. It means that $z_* \in C$.

Theorem is proved now.

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