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A Remark on the Existence of Positive Solution to a Nonlinear Semipositone System Involving the Weight Function

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Abstract

In this note we are mainly concerned with existence result for semilinear semipositone elliptic equation of the form

$$\begin{cases} -\Delta u = \lambda a(x) v^{\alpha} - c, & x \in \Omega, \\ -\Delta v = \lambda b(x) u^{\beta} - c, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where Δ denote the Laplacian operator, Ω is a bounded domain in $R^N (N > 1)$ with $\partial\Omega$ of class C^2, λ, c are positive parameters, α, $\beta > 0$ and the weight $a(x)$, $b(x)$ satisfying $a(x) \in C(\Omega)$, $b(x) \in C(\Omega)$ and $a(x) \geq a_0 > 0$, $b(x) \geq a_0 > 0$ for $x \in \Omega$. We prove the existence of positive solution via the method of sub-super solutions.

Mathematics Subject Classification: 35J55

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1 Introduction

In this paper we consider the existence of positive solution for the following nonlinear semipositone system

$$\begin{cases} -\Delta u = \lambda a(x)v^{\alpha} - c, & x \in \Omega, \\ -\Delta v = \lambda b(x)u^{\beta} - c, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δ denote the Laplacian operator, Ω is a bounded domain in $R^N (N > 1)$ with $\partial\Omega$ of class C^2, λ, c are positive parameters and α, $\beta > 0$.

In recent years, many authors have investigated the following initial boundary value problem of a class of reaction-diffusion system

$$\begin{cases} u_t = \Delta u + v^{\alpha}, \\ v_t = \Delta v + u^{\beta}, \qquad (x, t) \in \Omega \times (0, T), \end{cases} \tag{2}$$

where Ω is as above. Yang and Lu [17] studied the nonexistence of positive solutions to the system (2).

When $c = 0$, systems of the form (1) arise in several context in biology and engineering (see [13]). It provides a simple model to describe, for instance, the interaction of three diffusing biological species. u, v represent the densities of three species. See [16] for details on the physical models involving more general reaction-diffusion system.

Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [1, 2, 3, 14, 15]). We refer to [6, 8] for additional results in semipositone problems. See [5], where the author investigated the problem (1) in the case when $a(x) \equiv 1$, $b(x) \equiv 1$. Our purpose in this paper to study the problem (1) with weight. Our approach is based on the method of sub-super solutions, see [10]. We refer to [4, 7, 9, 12] for additional results on elliptic systems.

In this paper, we shall prove that if $\alpha, \beta < 1$ then there exist positive constant c_0 and λ^* such that (1) admits a positive solution for for $c \leq c_0$ and $\lambda \geq \lambda^*$.

2 Main results

To prove our existence results we use the method of sub-super solutions. To do so, we first give the definition of sub-super solution of (1).

Definition 2.1. A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) in $C_0^2(\bar{\Omega}) \times C_0^2(\bar{\Omega})$ are called a subsolution and supersolution of (1) if they satisfy $\psi_i(x) \leq z_i(x)$ in Ω for $i = 1, 2$, and

$$-\Delta\psi_1 \leq \lambda a(x)\,\psi_2^{\alpha} - c, \;\; -\Delta\psi_2 \leq \lambda b(x)\,\psi_1^{\beta} - c, \qquad x \in \Omega,$$

and

$$-\Delta z_1 \geq \lambda\; a(x) z_2^{\alpha} - c, \;\; -\Delta z_2 \geq \lambda\; b(x) z_1^{\beta} - c, \qquad x \in \Omega.$$

We shall obtain the existence of positive solution to system (1) by constructing a positive subsolution (ψ_1, ψ_2) and supersolution (z_1, z_2). It is well known that if there exists a sub solution (ψ_1, ψ_2) and a super solution (z_1, z_2) to (1) such that $\psi_i(x) \leq z_i(x)$ for $i = 1, 2$ and $x \in \bar{\Omega}$, then (1) has a solution (u, v) such that $\psi_1(x) \leq u(x) \leq z_1(x)$ and $\psi_2(x) \leq v(x) \leq z_2(x)$ for $x \in \bar{\Omega}$. Further note that if $\psi_i(x) \geq 0$ for $i = 1, 2$ and $x \in \Omega$ then $u \geq 0$ and $v \geq 0$ for $x \in \Omega$.

Our main result is formulate in the following theorem.

Theorem 2.2. Let $0 < \alpha < 1$, $0 < \beta < 1$, then there exist positive constants c_0 and λ^* such that (1) has a positive solution for $c \leq c_0$ and $\lambda \geq \lambda^*$.

Proof. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 denote the corresponding eigenfunction, satisfying $\phi_1(x) > 0$ in Ω, $|\nabla\phi_1| > 0$ on $\partial\Omega$ and $||\phi_1||_\infty = 1$. To obtain the existence of positive solution to problem (1), we constructing a positive subsolution (ψ_1, ψ_2) and supersolution (z_1, z_2). We shall verify that $(\psi_1, \psi_2) = (\psi, \psi)$, where $\psi = \frac{1}{2}\phi_1^2$, is a subsolution of (1). A calculation shows that

$$\begin{aligned} -\Delta\psi &= -\frac{1}{2}\,\Delta\phi_1^2 \\ &= -\left(|\nabla\phi_1|^2 + \phi_1\,\Delta\phi_1\right) \\ &= \lambda_1\phi_1^2 - |\nabla\phi_1|^2. \end{aligned}$$

Which implies thThen ψ is a subsolution if

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le \lambda\, a(x)\, \psi^\alpha - c,$$

and

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le \lambda\, b(x)\psi^\beta - c.$$

Since $\phi_1 = 0$ and $|\nabla\phi_1| > 0$ on $\partial\Omega$, there exist positive constants ϵ, δ, η such that

$$\lambda_1\, \phi_1^2 - |\nabla\phi_1|^2 \le -\epsilon, \qquad x \in \bar{\Omega}_\delta, \tag{3}$$

$$\phi_1 \ge \eta, \qquad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta, \tag{4}$$

with $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. Now $\lambda_1\, \phi_1^2 - |\nabla\phi_1|^2 \le -\epsilon$ in $\bar{\Omega}_\delta$, and therefore
if $c \le c_0 = \min\{\epsilon, \lambda\, a_0\, (\eta^2/2)^\alpha - \lambda_1, \lambda\, b_0\, (\eta^2/2)^\beta - \lambda_1\}$, then

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le \lambda\, a(x)\psi^\alpha - c,$$

and

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le \lambda\, b(x)\, \psi^\beta - c.$$

Next, we note that $\phi_1(x) \ge \eta > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\delta$ for some $\eta > 0$. If $c \le c_0 = \min\{\epsilon, \lambda\, a_0(\eta^2/2)^\alpha - \lambda_1, \lambda\, b_0(\eta^2/2)^\beta - \lambda_1\}$ and $\lambda \ge \lambda^* = \max\{(\frac{2}{\eta^2})^\alpha \frac{\lambda_1}{a_0}, (\frac{2}{\eta^2})^\beta \frac{\lambda_1}{b_0}\}$, then we have

$$\begin{aligned}
\lambda_1\phi_1^2 - |\nabla\phi_1|^2 &\le \lambda\, a_0\, (\frac{1}{2})^\alpha\, \eta^{2\alpha} - c \\
&\le \lambda\, a(x)\, (\frac{1}{2})^\alpha\, \phi_1^{2\alpha} - c \\
&= \lambda\, a(x)\, \psi^\alpha - c, \qquad x \in \Omega_0,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1\phi_1^2 - |\nabla\phi_1|^2 &\le \lambda\, b_0\, (\frac{1}{2})^\beta\, \eta^{2\beta} - c \\
&\le \lambda\, b(x)\, (\frac{1}{2})^\beta\, \phi_1^{2\beta} - c \\
&= \lambda\, b(x)\, \psi^\beta - c, \qquad x \in \Omega_0.
\end{aligned}$$

Hence

$$\begin{aligned}
-\Delta\psi &= \lambda_1\phi_1^2 - |\nabla\phi_1|^2 \\
&\le \lambda\, a(x)\, \psi^\alpha - c, \qquad x \in \Omega_0,
\end{aligned}$$

and

$$\begin{aligned} -\Delta\psi &= \lambda_1 \phi_1^2 - |\nabla\phi_1|^2 \\ &\leq \lambda\, b(x)\, \psi^\beta - c, \qquad x \in \Omega_0. \end{aligned}$$

i.e. (ψ, ψ) is a subsolution of (1).

Next, we consider the unique solution, $e(x) \in C^1(\overline{\Omega})$, of the boundary value problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $e(x) > 0$ in Ω and $\frac{\partial e(x)}{\partial n} < 0$ on $\partial\Omega$. We construct a supersolution (z_1, z_2) of (1). We denote $(z_1, z_2) = (A_1\, e(x), A_2\, e(x))$ where the constant A_1, $A_2 > 0$ are large and to be chosen later. We shall verify that (z_1, z_2) is a supersolution of (1). A calculation shows that

$$-\Delta z_1 = A_1, \qquad -\Delta z_2 = A_2.$$

Let $l = ||e(x)||_\infty$, it is easy to prove that there exist positive large constants A_1, A_2 such that

$$A_1 \geq \lambda\, \|a\|_\infty\, (A_2\, l)^\alpha, \qquad A_2 \geq \lambda\, \|b\|_\infty\, (A_1\, l)^\beta.$$

Then we have

$$\begin{aligned} A_1 &\geq \lambda \|a\|_\infty\, (A_2\, l)^\alpha \\ &\geq \lambda\, a(x)\, (A_2\, l)^\alpha - c \\ &\geq \lambda\, a(x)\, (A_2\, e(x))^\alpha - c \\ &= \lambda\, a(x)\, (z_2)^\alpha - c. \end{aligned}$$

Similarly we have

$$\begin{aligned} A_2 &\geq \lambda\, \|b\|_\infty\, (A_1\, l)^\beta \\ &\geq \lambda\, b(x)\, (A_1\, l)^\beta - c \\ &\geq \lambda\, b(x)\, (A_1\, e(x))^\beta - c \\ &= \lambda\, b(x)\, (z_1)^\beta - c. \end{aligned}$$

and therefore

$$-\Delta z_1 \geq \lambda\, a(x)\, z_2^{\alpha} - c, \quad -\Delta z_2 \geq \lambda\, b(x)\, z_1^{\beta} - c.$$

i.e. (z_1, z_2) is a supersolution of (1) with $z_i \geq \psi_i$ in Ω for large $A_1,\ A_2,\ i = 1, 2$. Thus, by the comparison principle, there exists a solution (u, v) of (1) with $\psi_1 \leq u \leq z_1, \quad \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 2.2. □

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