Valuation of an Option Strategy Under Uncertain Stock Models with Two-Way Jumps

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Abstract

Options are flexible derivatives which can be used in combination with underlying asset or other types of options to form specific option trading strategies that respond to the needs of different investors in different market environments. Considering that the price of the underlying asset is subject to both positive and negative shocks from certain factors, we use uncertain differential equations with positive jumps and negative jumps to describe the price changes. In this paper, we value a widely used trading strategy called bull call spread and analyze its monotonicity with respect to each factor.

Keywords: Option trading strategies, Positive jumps and negative jumps, Uncertain differential equations

1 Introduction

Options are contracts that give the holder the right to purchase or sell an asset at a fixed price on a specific date or at any time prior to that date. The core problem of stock option pricing is to find a suitable process to better describe the pattern of stock price movement. The earliest research on option pricing dates back to 1900, when Louis Bachelier first proposed an option pricing model in his Ph.D. thesis and used Brownian motion to predict stock price movements. Black and Scholes [8] used the Ito formula [1] in 1973 to obtain the pricing formula for European options based on the theory of stochastic analysis,
while Merton [7] independently established the theory of option pricing. They developed what became known as the Black-Scholes-Merton (BSM formula) model, which has had a significant impact on how traders price and hedge options.

The traditional theory of financial decision making is based on the framework of probability theory. However, empirical studies and the development of behavioral finance have questioned the use of probability theory for research. Furthermore, it can be seen that people’s confidence plays a very important role in behavioral decision making. In order to deal with the confidence problem in a reasonable way, Liu created the uncertainty theory [3] in 2007 and gave an uncertain stock model [5] in 2009 under the assumption that stock prices obey the geometric canonical process. In 2012, Yu [9] proposed an uncertain stock model with jumps, which added a jump term to the original uncertain stock model. Ji and Zhou [2] in 2015 also subdivided this jump into the positive jump as well as the negative jump, and proposed an uncertain stock model with two-way jumps, which can better fit stock movement patterns in the real world.

Options are often used in combination with underlying asset to form specific options trading strategies to address the needs of different investors in different market environments. The vertical spread strategy, which simultaneously buys and sells options with the same expiration time but different strike prices, is one of the widely used option trading strategies. The vertical spread strategy includes bull spread, bear spread, and butterfly spread.

In this paper, we require into the valuation of bull call spread strategy. We use the uncertain stock model with two-way jumps to give the valuation formula for the bull call spread strategy. Then, we give their monotonicity with respect to each factor and numerical examples of strategies.

2 Preliminaries

An uncertain variable [3] is a measurable function from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real number, that is, for any Borel set B of real number, the set \(\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}\) is an event in \(\mathcal{L}\). To describe an uncertain variable \(\xi\) in practice, a concept of uncertainty distribution was defined by Liu as \(\Phi (x) = \mathcal{M}\{\xi \leq x\}\) for any real number \(x\). Let \(\xi\) be an uncertain variable with an uncertainty distribution \(\Phi\). Then the expected value of \(\xi\) is defined by

\[
E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} \, dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} \, dx,
\]

provided that at least one of the integrals is finite.
Theorem 2.1. ([6]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_m$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is continuous, strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n) = x} \left( \min_{1 \leq i \leq m} \Phi_i(x_i) \land \min_{m+1 \leq i \leq n} (1 - \Phi_i(x_i)) \right).$$

An uncertain process $C_t$ is said to be a Liu process [5] if (i) $C_0$ and almost all sample paths are Lipschitz continuous, (ii) $C_t$ has stationary and independent increments, (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Psi_t(x) = \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3} \sigma} \right) \right)^{-1}, x \in \mathbb{R}. \quad (2)$$

Let $\xi_1, \xi_2, \cdots$ be iid positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N_t = \max_{n \geq 0} \{ n | S_n \leq t \} \quad (3)$$

is called an uncertain renewal process [4].

Let $X_t$ be a bond price, and $Y_t$ be a stock price. Assume that price $Y_t$ follows an uncertain differential equation driven by a canonical Liu process $C_t$ and uncertain renewal processes $N_{1t}$ and $N_{2t}$. Then Ji and Zhou [2] proposed an uncertain stock model as follows,

$$\begin{cases} 
  dX_t = r X_t dt. \\
  dY_t = \mu Y_t dt + \sigma Y_t dC_t + \nu_1 Y_t dN_{1t} + \nu_2 Y_t dN_{2t}.
\end{cases} \quad (4)$$

where $r$ is the riskless interest rate, $\mu$ is the stock drift, $\sigma > 0$ is the stock diffusion, $\nu_1 > 0$, $-1 < \nu_2 < 0$, $N_{1t}$ is the positive jump process and $N_{2t}$ is the negative jump process. Furthermore, assume that $C_t, N_{1t}, N_{2t}$ are independent.

3 Bull call spread strategy

Bull spread strategy gains when the underlying asset market rises moderately, but the maximum gain and maximum loss are both limited. Therefore, when investors expect the asset price to rise moderately, they can adopt a bull spread strategy to gain moderately while ensuring that both losses and gains are within certain limits. Depending on the type of option used, bull spread strategy can be divided into bull call spread strategy as well as bull put spread strategy.
Bull call spread strategy is traded by buying a call option with a lower strike price and selling a call option with the same expiration date but a higher strike price, and is more suitable for situations with low volatility.

We assume that the call option with a lower strike price has an exercise price of \( K_1 \) and a purchase price of \( C_1 \), while the call option with a higher exercise price has an exercise price of \( K_2 \) and a sell price of \( C_2 \). Obviously, we can get that \( K_2 > K_1 \) and \( C_1 > C_2 \).

Denote the value of the spread at time 0 by \( V_{BUC} \) and the payoff of the option at time \( T \) by \( P_T \). This payoff contains the returns of the two call options at time \( T \), which are \( (Y_T - K_1)^+ \) and \( -(Y_T - K_2)^+ \), respectively. In summary, at moment \( T \), the combined return of the strategy is

\[
P_T = (Y_T - K_1)^+ - (Y_T - K_2)^+ = \begin{cases} 
0, & \text{if } Y_T < K_1 \\
Y_T - K_1, & \text{if } K_1 \leq Y_T \leq K_2 \\
K_2 - K_1, & \text{if } Y_T > K_2.
\end{cases}
\]

The value of the spread at time 0 is \( V_{BUC} = C_2 - C_1 + \exp(-rT)E[P_T] \).

**Theorem 3.1.** Assume that the stock price \( Y_T \) has an uncertainty distribution \( \Phi_T(x) \). Then

\[
\Phi_T(x) = \sup_{abc=x/Y_0} \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \wedge \Psi_T \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \wedge \left( 1 + \Psi_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right). 
\]  

(5)

**Proof** It follows from (4) that at time \( T \), we have

\[
Y_T = Y_0 \cdot \exp(\mu T + \sigma C_T) \cdot (1 + \nu_1)^{N_{1T}} \cdot (1 + \nu_2)^{N_{2T}}.
\]  

(6)

Thus

\[
\Phi_T(x) = \mathcal{M} \{ Y_T \leq x \} = \mathcal{M} \{ Y_0 \cdot \exp(\mu T + \sigma C_T) \cdot (1 + \nu_1)^{N_{1T}} \cdot (1 + \nu_2)^{N_{2T}} \leq x \} = \mathcal{M} \{ \exp(\mu T + \sigma C_T) \cdot (1 + \nu_1)^{N_{1T}} \cdot (1 + \nu_2)^{N_{2T}} \leq x/Y_0 \}. 
\]  

(7)

The function \( f = \exp(\mu T + \sigma C_T) \cdot (1 + \nu_1)^{N_{1T}} \cdot (1 + \nu_2)^{N_{2T}} \) is monotonically increasing with respect to \( \exp(\mu T + \sigma C_T) \), \( (1 + \nu_1)^{N_{1T}} \), \( (1 + \nu_2)^{N_{2T}} \), and \( (1 + \nu_1)^{N_{1T}} \), \( (1 + \nu_2)^{N_{2T}} \), \( x/Y_0 \) are all positive. Suppose that \( a, b, c \) are positive numbers and \( abc = x/Y_0 \). Then according to Theorem 2.1 it follows that

\[
\Phi_T(x) = \sup_{abc=x/Y_0} \mathcal{M} \{ (\exp(\mu T + \sigma C_T) \leq a) \} \wedge \mathcal{M} \{ (1 + \nu_1)^{N_{1T}} \leq b \} \wedge \mathcal{M} \{ (1 + \nu_2)^{N_{2T}} \leq c \} = \sup_{abc=x/Y_0} \mathcal{M} \{ \mu T + \sigma C_T \leq \ln a \} \wedge \mathcal{M} \{ N_{1T} \cdot \ln(1 + \nu_1) \leq \ln b \} \wedge \mathcal{M} \{ N_{2T} \cdot \ln(1 + \nu_2) \leq \ln c \}. 
\]  

(8)
As \( \ln(1 + \nu_1) > 0, \ln(1 + \nu_2) < 0 \) and \( \sigma > 0 \), we can get
\[
\Phi_T(x) = \sup_{abc=x/Y_0} \mathcal{M} \left\{ C_T \leq \frac{\ln a - \mu T}{\sigma} \right\} \wedge \mathcal{M} \left\{ N_{1T} \leq \frac{\ln b}{\ln(1 + \nu_1)} \right\} \wedge \mathcal{M} \left\{ N_{2T} \geq \frac{\ln c}{\ln(1 + \nu_2)} \right\}.
\]
(9)

Let \( \Psi_T, \Upsilon_{1T}, \Upsilon_{2T} \) be the distributions of \( C_T, N_{1T}, N_{2T} \), respectively. We get
\[
\Phi_T(x) = \sup_{abc=x/Y_0} \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \wedge \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \wedge \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right).
\]
(10)

**Theorem 3.2. (Bull call spread strategy valuation formula)** The value of a bull call spread strategy in the uncertain market is
\[
V_{BUC} = C_2 - C_1 + \exp(-rT) \int_{K_1}^{K_2} \inf_{abc=x/Y_0} \left( 1 - \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right) \wedge \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \wedge \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right) dx,
\]
(11)
where \( \Psi_t, \Upsilon_{1t} \) and \( \Upsilon_{2t} \) are the uncertainty distributions of \( C_t, N_{1t} \) and \( N_{2t} \), respectively.

**Proof** Suppose \( \varphi(x) \) is the distribution of \( P_T \). Since \( P_T \) is increasing with respect to \( Y_T \), according to Theorem 2.1, we get
\[
\varphi(x) = \sup_{(x_1-K_1)\text{+}-(x_1-K_2)\text{=}x} \Phi_T(x_1),
\]
where
\[
x = (x_1-K_1)\text{+}-(x_1-K_2)\text{=}x = \begin{cases} \ 0, & \text{if } x_1 < K_1 \\ x_1 - K_1, & \text{if } K_1 \leq x_1 \leq K_2 \\ K_2 - K_1, & \text{if } x_1 > K_2. \end{cases}
\]

According to (1), we have
\[
E[P_T] = \int_{0}^{+\infty} (1 - \varphi(x)) dx - \int_{-\infty}^{0} \varphi(x) dx = \int_{0}^{K_2-K_1} (1 - \varphi(x)) dx.
\]
Since the range of integration of \( x \) is \((0, K_2 - K_1)\), and \( x = x_1 - K_1 \), we have
\[
\varphi(x) = \sup_{(x_1-K_1)\text{+}-(x_1-K_2)\text{=}x} \Phi_T(x_1) = \Phi_T(x + K_1).
\]
Thus

\[
E[P_T] = \int_0^{K_2 - K_1} (1 - \Phi_T(x + K_1)) dx
= \int_{K_1}^{K_2} (1 - \Phi_T(x)) dx
= \int_{K_1}^{K_2} \left( 1 - \sup_{abc=x/Y_0} \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right)
\wedge \left( 1 - \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \right)
\wedge \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right) dx
= \int_{K_1}^{K_2} \inf_{abc=x/Y_0} \left( 1 - \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right)
\vee \left( 1 - \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \right)
\vee \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right) dx. \tag{12}
\]

The present value of the portfolio returns after discounting is

\[
\exp(-rT)E[P_T] = \exp(-rT) \int_{K_1}^{K_2} \inf_{abc=x/Y_0} \left( 1 - \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right)
\vee \left( 1 - \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \right)
\vee \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right) dx. \tag{13}
\]

Because the call option with the lower strike price at the beginning of the period was bought at \( C_1 \) and the call option with the higher strike price was sold at \( C_2 \), the value of the bull call spread strategy is

\[
V_{BUC} = C_2 - C_1 + \exp(-rT) \int_{K_1}^{K_2} \inf_{abc=x/Y_0} \left( 1 - \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right)
\vee \left( 1 - \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \right)
\vee \left( 1 - \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right) \right) dx.
\]

**Theorem 3.3.** As the function of \( Y_0, r, K_1, K_2, C_1, C_2, \mu, \sigma, \nu_1, \nu_2 \), bull call spread strategy valuation \( V_{BUC} = C_2 - C_1 + \exp(-rT)E[P_T] \) has the following properties when \( K_1 \leq Y_T \leq K_2 \):

(i) \( V_{BUC} \) is a decreasing function of \( Y_0 \);
(ii) \( V_{BUC} \) is a decreasing function of \( r \);
(iii) \( V_{BUC} \) is a decreasing function of \( K_1 \) and a increasing function of \( K_2 \);
(iv) \( V_{BUC} \) is a decreasing function of \( C_1 \) and a increasing function of \( C_2 \);
(v) \( V_{BUC} \) is a increasing function of \( \mu \);
(vi) \( V_{BUC} \) is a increasing function of \( \sigma \);
(iv) \( V_{BUC} \) is a increasing function of \( \nu_1 \) and a decreasing function of \( \nu_2 \).
Proof (i) It is obvious that $V_{BUC} = C_2 - C_1 + \exp(-rT)E[Y_0 \cdot \exp(\mu T + \sigma C_T) \cdot (1 + \nu_1)^{N_{1T}} \cdot (1 + \nu_2)^{N_{2T}} - K_1]$ is an increasing function with respect to $Y_0$. This indicates that the value of a bull call option strategy grows as the initial stock price rises.

(ii) The expected value $E[Y_T - K_1]$ is greater than 0 and independent of $r$. So $V_{BUC} = C_2 - C_1 + \exp(-rT)E[P_T]$ is a decreasing function of $r$. This means the value of the bull call spread strategy will decrease with the riskless interest rate.

(iii) The integrand in (11)

$$\inf_{abc=x/Y_0} \left( 1 - \Psi_T \left( \frac{\ln a - \mu T}{\sigma} \right) \right) \vee \left( 1 - \Upsilon_{1T} \left( \frac{\ln b}{\ln(1 + \nu_1)} \right) \right) \vee \Upsilon_{2T} \left( \frac{\ln c}{\ln(1 + \nu_2)} \right)$$

is always nonnegative for each $x > 0$. As the strike price $K_1$ increases, the domain $[K_1, K_2]$ of the integration decreases, which makes the integral becomes smaller. Thus $V_{BC}$ decreases with respect to the strike price $K_1$; As the strike price $K_2$ increases, the domain $[K_1, K_2]$ of the integration increases, which makes the integral becomes bigger. Thus $V_{BUC}$ increases with respect to the strike price $K_2$.

(iv) It is obvious that $V_{BUC} = C_2 - C_1 + \exp(-rT)E[P_T]$ decreases as $C_1$ increases and increases as $C_2$ increases.

(v) It is clear that $V_{BUC} = C_2 - C_1 + \exp(-rT)E[Y_0 \cdot \exp(\mu t + \sigma C_t) \cdot (1 + \nu_1)^{N_{1t}} \cdot (1 + \nu_2)^{N_{2t}} - K_1]$ is an increasing function of $\mu$. This means that the value of the bull call spread strategy will increase with the stock drift.

(vi) It is clear that $V_{BUC} = C_2 - C_1 + \exp(-rT)E[Y_0 \cdot \exp(\mu t + \sigma C_t) \cdot (1 + \nu_1)^{N_{1t}} \cdot (1 + \nu_2)^{N_{2t}} - K_1]$ is an increasing function of $\sigma$. It means that the value of the bull call spread strategy will increase with the stock diffusion.

(vii) It is obvious that $V_{BUC} = C_2 - C_1 + \exp(-rT)E[Y_0 \cdot \exp(\mu t + \sigma C_t) \cdot (1 + \nu_1)^{N_{1t}} \cdot (1 + \nu_2)^{N_{2t}} - K_1]$ is an increasing function of $\nu_1$ as $\nu_1 > 0$, and a decreasing function of $\nu_2$ as $-1 < \nu_2 < 0$.

Example 3.4. Assume that the uncertain stock model (4) has the following parameters: risk-free rate $r = 0.05$, drift coefficient $\mu = 0.06$, volatility coefficient $\sigma = 0.3$, positive jump coefficient $\nu_1 = 1.1$, and negative jump coefficient $\nu_2 = -0.091$. Consider a bull call option strategy with a stock opening price $Y_0 = 20$, a bought call option strike price $K_1 = 25$, expiration date $T = 2$ and bid price $C_1 = 6$, and a sold call option strike price $K_2 = 30$, expiration date $T = 2$ and ask price $C_2 = 5$. According to valuation formula (11), the calculation is performed and the value of the bull call option strategy is obtained to be $V_{BUC} = 3.3387$. 
4 Conclusion

Uncertain stock models with two-way jumps is a stock model based on uncertain differential equation, taking into account both positive and negative market shocks, which can portray the stock movement patterns more realistically. The bull call spread is a kind of widely used option strategy, and in this paper, we obtained a valuation formula of option strategy for the stock model. Furthermore, we analyze the impact of various factors such as initial stock price, stock drift and stock diffusion on the valuation.

References


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