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Multiobjective Programming Problems with (G, C, α, ρ, d)-Convexity

Xiaoling Liu

Department of Mathematics, Hanshan Normal University
Chaozhou, 521041 China

Dehui Yuan

Department of Mathematics, Hanshan Normal University
Chaozhou, 521041 China

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Abstract

In this paper, we present necessary optimality conditions for multiobjective programming problems involving functions which is not necessarily differential. A new concept of generalized convexity, which is called (G, C, α, ρ, d)-convexity, is introduced. We also establish sufficient optimality conditions for multiobjective programming problems from a viewpoint of the new generalized convexity. When the sufficient conditions are utilized, the corresponding duality theorems are derived for general Mond-Weir type dual program.

Keywords: multiobjective program, (G, C, α, ρ, d)-convex, G-necessary conditions, G-sufficient conditions, G-duality theory

1 Introduction

It is well known that convexity has been playing a key role in mathematical programming, engineering and optimization theory. The research on characterizations and generalizations of convexity is one of the most important aspects in mathematical programming and optimization theory. And many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature.

To relax convexity assumptions imposed on theorems on sufficient optimality conditions and duality, various generalized convexity notations have been proposed. Hanson [10] introduced the concept of differentiable invexity which is a generalization of the convexity. After the works of Hanson, other classes of differentiable nonconvex functions have appeared with the intent of generalizing the class of invex functions from different points of view in the literature [5, 8, 11, 17, 13, 19, 6, 12].

In [14] and [15], Liang et al. introduced (F, α, ρ, d)-convexity, which was a unified formulation of generalized convexity and which is an extension of (F, ρ)-convexity [19] and generalized (F, ρ)-convexity [6]. They obtained some corresponding optimality conditions and duality results for the single objective fractional problems and multiobjective problems. In a recent paper [20], Yuan et al. introduced (C, α, ρ, d)-convexity, which is a generalization of (F, α, ρ, d)-convexity. Chinchuluun et al. [7] and Long [16]later studied multiobjective fractional programming problems in the framework of (C, α, ρ, d)-convexity.

Recently, Antczak extended further Hanson's invexity to G-invexity for scalar differentiable functions[2]. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions in [3]. Using the G-invexity as a generalization of invexity in the vectorial case, Antczak[3, 4] established some optimality and duality results for a larger class of smooth multiobjective programming problems than invex vector optimization problems.

In this paper, we are motivated by Yuan et al. [20, 21, 22], Antczak [2, 3, 4] to consider optimality conditions and duality theorems for the general multiobjective programming in the framework of a new generalized convexity, which is called (G, C, α, ρ, d)-convexity. The paper is organized as follows. The formulation of the general multiobjective programming problem along with some definitions and notations for generalized convexity are given in Section 2. In Section 3, we obtain the necessary optimality conditions for general multiobjective programming problems under some assumptions. We also establish sufficient conditions for general multiobjective programming problems involving (G, C, α, ρ, d)-convex functions in section 3. When the sufficient conditions are utilized, dual problem is formulated and duality results are presented in Section 4.

2 Notations and Preliminaries

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$x > y$ if and only if $x_i > y_i$, for $i = 1, 2, \cdots, n$;
$x \geqslant y$ if and only if $x_i \geq y_i$, for $i = 1, 2, \cdots, n$;
$x \geq y$ if and only if $x_i \geq y_i$, for $i = 1, 2, \cdots, n$, but $x \neq y$;
$x \not> y$ is the negation of $x > y$, $x \not\geq y$ is the negation of $x \geq y$.

Let $R^n_+ = \{x \in R^n | x \geqslant 0\}$ and X be a subset of R^n.

Definition 2.1. *A function $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is convex on R^n with respect to the third argument if and only if, for any fixed $(x, x_0) \in X \times X$, the inequality*

$$C_{(x,x_0)}(\lambda\tau_1 + (1-\lambda)\tau_2) \leqslant \lambda C_{(x,x_0)}(\tau_1) + (1-\lambda)C_{(x,x_0)}(\tau_2),\ \forall\, \lambda \in (0,1)$$

holds for all $\tau_1, \tau_2 \in R^n$.

Throughout this paper, we assume that $C_{(x,x_0)}(0) = 0$ for any $(x, x_0) \in X \times X$. Now we introduce a generalized convexity based on the convex functions $C_{(x,x_0)}$ as follows:

Definition 2.2. *Let $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ be a vector-valued function defined on a nonempty set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$ be the range of $f_i, i \in K = \{1, \cdots, k\}$. If there exist a vector-valued function $G_f = (G_{f_1}, \cdots, G_{f_k}) : R \to \mathbb{R}^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain and $G_{f_i}(f_i)$ is a differentiable function on X. If there exist real numbers $\rho_i (i \in K)$, real-valued functions $\alpha_i : X \times X \to \mathbb{R}_+ \backslash \{0\} (i \in K)$, and $d_i : X \times X \to R_+ (i \in K)$ be functions with the property that $d_i(x, x_0) = 0 \Leftrightarrow x = x_0$, such that for any $x \in X (x \neq x_0)$, the inequality*

$$\frac{G_{f_i}(f_i(x)) - G_{f_i}(f_i(x_0))}{\alpha_i(x,\, x_0)} \geqslant (>) C_{(x,\, x_0)}\left(\nabla(G_{f_i}(f_i))(x_0)\right) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x,\, x_0)}$$

holds for each $i \in K$, then f is said to be (strictly) $(G_f, C, \alpha, \rho, d)$-convex at $x_0 \in X$, where $\alpha = (\alpha_1, \cdots, \alpha_k)^T$, $\rho = (\rho_1, \cdots, \rho_k)^T$ and $d = (d_1, \cdots, d_k)^T$. The function f is said to be (strictly) $(G_f, C, \alpha, \rho, d)$-convex over X if, $\forall\, x_0 \in X$, it is (strictly) $(G_f, C, \alpha, \rho, d)$-convex at x_0. In particular, f is said to be strongly (strictly) $(G_f, C, \alpha, \rho, d)$-convex or (strictly) (G_f, C, α)-convex with respect to $\rho > 0$ or $\rho = 0$, respectively.

Remark 2.3. *In order to define an analogous class of (strictly) $(G_f, C, \alpha, \rho, d)$-incave functions, the function G_{f_i} of the inequality in the above definition should be replaced by the function $-G_{f_i}$. That is the inequality*

$$\frac{-(G_{f_i}(f_i(x)) - G_{f_i}(f_i(x_0)))}{\alpha_i(x,\, x_0)} \geqslant (>) C_{(x,\, x_0)}\left(-\nabla(G_{f_i}(f_i))(x_0)\right) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x,\, x_0)}$$

holds for each $i \in K$

From Definition 2.2, $(G_f, C, \alpha, \rho, d)$-convexityis is (C, α, ρ, d)-convexity [20] whenever $G_f(\tau) = \tau, \tau \in \mathbb{R}$; Therefore (F, α, ρ, d)-convexity [14, 15] and (F, ρ)-convexity [19] are special cases of $(G_f, C, \alpha, \rho, d)$-convexity since any linear function is also a convex function. However, the converse result is, in general, not true (see Example 2.1)

Example 2.4. *Let $f(x) = \log x$. Since f is a well-known concave function defined on $X = \{x : x \in \mathbb{R}, x > 0\}$, then f is not a (C, α, ρ, d)-convex function. However, $G_f(f(x)) = x$ is convex function on X, where $G_f(\tau) = e^{\tau}, \tau \in \mathbb{R}$. Therefore, by Definition 2.2, f is a $(G_f, C, \alpha, \rho, d)$-convex function.*

Let f be differentiable on X and G_f be differentiable on its domain $I_f(X)$. It is clear that every G-invex with respect to vector function η [2] is $(G_f, C, \alpha, \rho, d)$-convex function with the same function G_f, where $d(x, x_0) = \|x - x_0\|$,$\alpha(x, x_0) \equiv 1$, $\rho = 0$ and

$$C_{(x,x_0)}\left(\nabla(G_{f_i}(f_i))(x_0)\right) = \left(G'_f(f(x_0))\nabla f(x_0)\right)^T \eta(x, x_0).$$

Therefore, r-invex functions [1] are $(G_f, C, \alpha, \rho, d)$-convex with the same function G_f, where $G_f(\tau) = e^{r\tau}, \tau \in \mathbb{R}$, because r-invex function is a special case of G-invex function. However, $(G_f, C, \alpha, \rho, d)$-convex function is not always G-invex, and we can see the next Example 2.2.

Example 2.5. *Let $f(x) = \sqrt{|x - 1|}$ be a function defined on $\mathbb{R}$. It is clear that f is not G-invex at $x^* = 1$, because f is not differentiable at $x^* = 1$. However, let $G_f(\tau) = \tau^3, \tau \in \mathbb{R}$, then $G_f(f)$ is differentiable at $x^* = 1$, and we can see that f is (G, C, α, ρ, d)-convex at $x^* = 1$. In fact, note that*

$$G_f(f(x)) \geq 0 = G_f(f(x^*)) and \nabla(G_f(f))(x^*) = 0,$$

we obtain

$$C_{(x,x^*)}(\nabla(G_f(f))(x^*)) := \nabla(G_f(f))(x^*)^T \eta(x, x^*) \leq G_f(f(x)) - G_f(f(x^*)).$$

for any function $\eta(x, x^*)$. Hence f is $(G_f, C, \alpha, \rho, d)$-convex at $x^* = 1$ with $\alpha(x, x^*) \equiv 1$, $\rho = 0$ and any function d.

One of important feature of $(G_f, C, \alpha, \rho, d)$-convex functions is that they are convex transformable. In other words, they can be transformed into (C, α, ρ, d)-convex functions, see the above Example 2.4. Another important feature of $(G_f, C, \alpha, \rho, d)$-convex functions is that they can transform nondifferentiable functions into differentiable ones. Let us see the above Example 2.5.

In this sequel, we consider the nonlinear multi-objective programming problem

$$\begin{aligned}(CVP) \quad & \min f(x) := (f_1(x), f_2(x), \cdots, f_k(x)),\\ & s.t.\ g(x) := (g_1(x), g_2(x), \cdots, g_m(x)) \leqq 0\\ & h(x) := (h_1(x), h_2(x), \cdots, h_p(x)) = 0\\ & x \in E\end{aligned}$$

where X is a nonempty set of $\mathbb{R}^n$, and f_i denotes a real-valued function on X. We denote by $K := \{1, 2, \cdots, k\}$, $M := \{1, 2, \cdots, m\}$, $P := \{1, 2, \cdots, p\}$ and $J(x) = \{j \in M : g_j(x) = 0\}$. Further, X denote the set of all feasible points of (CVP).

For the convenience, we need the following vector minimization problem:

$$\begin{aligned}(G - CVP) \quad & \min G_f f(x) := (G_{f_1}(f_1(x), G_{f_2}(f_2(x)), \cdots, G_{f_k}(f_k(x)))\,,\\ & s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), \cdots, G_{g_m}(g_m(x))) \leqq G_g(0)\\ & G_h h(x) := \big(G_{h_1}(h_1(x)), \cdots, G_{h_p}(h_p(x))\big) = G_h(0)\\ & x \in E\end{aligned}$$

Before studying optimality in multiobjective programming, one has to define clearly the concepts of optimality and solutions in multiobjective programming problem. Note that, in vector optimization problems there is a multitude of competing definitions and approaches. The dominated ones are now various scalarizations and (weak) Pareto optimality. The (weak) Pareto optimality in multiobjective programming associates the concept of a solution with some property that seems intuitively natural.

Definition 2.6. *A feasible point $\bar{x}$ is said to be an efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in X$ such that*

$$f(x) \leq f(\bar{x}).$$

Definition 2.7. *A feasible point $\bar{x}$ is said to be a weakly efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in X$ such that*

$$f(x) < f(\bar{x}).$$

It is very easy to check the follow theorem, and we omitted the proof here.

Theorem 2.8. *Let $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Further, let $0 \in I_{g_j}(X)$, $j \in M$, and $0 \in I_{h_t}(X)$, $t \in M$. Then $\bar{x}$ is a (weak) efficient solution for (CVP) if and only if $\bar{x}$ is a (weak) efficient solution for (G-CVP).*

Definition 2.9. *Let X be a set of all feasible solutions in the multiobjective programming problem (G-CVP) and $\bar{x} \in X$. The multiobjective programming problem (G-CVP) is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x}$ if,*

$$C(X,\bar{x}) = \{d \in \mathbb{R}^n : \nabla(G_{g_j}(g_j))(\bar{x})d \leqq 0, j \in J(\bar{x}), \\ \nabla(G_{h_t}(h_t))(\bar{x})d = 0, t \in P\}$$

where $C(X,\bar{x})$ is the Bouligand tangent cone to X at $\bar{x}$, for details see [3].

3 Optimality conditions

In [2], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable mathematical programming problem. In a natural way, he extended the so-called G-Karush-Kuhn-Tucker necessary optimality conditions to the vectorial case [3] for differentiable multiobjective programming problems. In this section, we present a different G-Kuhn-Tucker necessary optimality conditions for multiobjective programming problems (CVP), in which each component of the objective function is not necessarily differential, through an auxiliary programming problem (G-CVP. Furthermore, we will prove the sufficiency of the introduced G-Karush-Kuhn-Tucker necessary optimality conditions under (G, C, α, ρ, d)-convexity defined in above section.

Theorem 3.1 (G-KKT necessary optimality conditions)**.** *Let G_{f_i} $(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Let $\bar{x}$ be a (weak) efficient solution for (CVP), $G_f(f)$, $G_g(g)$ and $G_h(h)$) be differentiable at $\bar{x}$. Moreover, we assume that the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_k)^T \in \mathbb{R}^k$, $\bar{\xi} = (\bar{\xi}_1, \cdots, \bar{\xi}_m)^T \in \mathbb{R}^m$ and $\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_p)^T \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla(G_{f_i}(f_i))(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j \nabla(G_{g_j}(g_j))(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t \nabla(G_{h_t}(h_t))(\bar{x}) = 0 \quad (1)$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0, \ j \in M, \forall x \in E \quad (2)$$

$$\bar{\lambda} \geq 0, \sum\nolimits_{i=1}^{k} \bar{\lambda}_i = 1, \bar{\xi} \geqq 0 \quad (3)$$

Proof. Since $\bar{x}$ is a (weak) efficient solution for (CVP), then, by Theorem 2.8, $\bar{x}$ is a (weak) efficient solution for (G-CVP). Therefore, the following system:

$$\begin{cases} \nabla(G_f(f))(\bar{x})d \leqq 0, \\ \nabla(G_g(g))(\bar{x})d \leqq 0, \\ \nabla(G_h(h))(\bar{x})d = 0 \end{cases}$$

is inconsistent. Hence, from From Motzkin's theorem [18], there exist $\lambda \in \mathbb{R}^k$, $\xi \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ such that

$$\sum_{i=1}^{k} \lambda_i \nabla (G_{f_i}(f_i))(\bar{x}) + \sum_{j=1}^{m} \xi_j \nabla (G_{g_j}(g_j))(\bar{x}) + \sum_{t=1}^{p} \mu_t \nabla (G_{h_t}(h_t))(\bar{x}) = 0 \quad (4)$$

$$\xi_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x})) \right) \leqslant 0,\ j \in M, \forall x \in E \quad (5)$$

$$\lambda \geq 0, \xi \geqq 0 \quad (6)$$

Multiplying (4) with $\lambda_0 = \frac{1}{\sum_{i=1}^{k} \lambda_i}$, we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_0} \nabla (G_{f_i}(f_i))(\bar{x}) + \sum_{j=1}^{m} \frac{\xi_j}{\lambda_0} \nabla (G_{g_j}(g_j))(\bar{x}) + \sum_{t=1}^{p} \frac{\mu_t}{\lambda_0} \nabla (G_{h_t}(h_t))(\bar{x}) = 0$$

Replacing $\frac{\lambda_i}{\lambda_0}$, $\frac{\xi_j}{\lambda_0}$ and $\frac{\mu_t}{\lambda_0}$ in the above equation by $\bar{\lambda}_i$, $\bar{\xi}_j$ and $\bar{\mu}_t$, respectively, and from (5) and (6), we can get the desired result. □

Theorem 3.2 (G-KKT sufficient optimality conditions)**.** *Let $\bar{x}$ be a feasible point for (CVP) such that G-Karush-Kuhn-Tucker necessary optimality conditions (1)-(3) are satisfied at $\bar{x}$. Assume that f is $(G_f, C, \alpha, \rho, d)$-convex at $\bar{x}$, $g_{J(\bar{x})}$ is $(G_{g_{J(\bar{x})}}, C, \beta, \rho', d')$-convex at $\bar{x}$, $h_t\,(t \in P^+)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-convex at $\bar{x}$, and $h_t\,(t \in P^-)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-concave at $\bar{x}$ on X, where $P^+ = \{t \in P : \bar{\mu}_t > 0\}$ and $P^- = \{t \in P : \bar{\mu}_t < 0\}$. If the following inequality*

$$\sum_{i=1}^{k} \bar{\lambda}_i \rho_i \frac{d_i(x, \bar{x})}{\alpha_i(x, \bar{x})} + \sum_{j=1}^{m} \bar{\xi}_j \rho'_j \frac{d'_j(x, \bar{x})}{\beta_j(x, \bar{x})} + \sum_{t=1}^{p} |\bar{\mu}_t| \bar{\rho}_t \frac{\bar{d}_t(x, \bar{x})}{\gamma_t(x, \bar{x})} \geqslant 0 \quad (7)$$

holds for any $x \in X$. Then $\bar{x}$ is a weakly efficient solution for (CVP).

Proof. Suppose, contrary to the result, that $\bar{x}$ is not a weakly efficient solution for (CVP). By Theorem 2.8, $\bar{x}$ is not a weakly efficient solution for (G-CVP). Hence, there exists $x_0 \in X$ such that

$$G_{f_i}(f_i(x_0)) < G_{f_i}(f_i(\bar{x})), i \in K. \quad (8)$$

Condition (2), (3), (8) and the fact that

$$g_j(x_0) \leqslant 0 = g_j(\bar{x}), j \in J(\bar{x}),$$
$$h_t(x_0) = 0 = h_t(\bar{x}), t \in P$$

can reduce the inequality

$$\sum_{i=1}^{k} \bar{\lambda}_i \frac{G_{f_i}(f_i(x_0)) - G_{f_i}(f_i(\bar{x}))}{\alpha_i(x_0, \bar{x})} + \sum_{j \in J(\bar{x})} \bar{\xi}_j \frac{G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x}))}{\beta_j(x_0, \bar{x})}$$
$$+ \sum_{t \in P^{+-}} |\bar{\mu}_t| \frac{G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))}{\gamma_t(x_0, \bar{x})} < 0 \quad (9)$$

where $P^{+-} = P^+ \cup P^-$. By the generalized invexity assumption of f, g and h, we have

$$\frac{G_{f_i}(f_i(x_0)) - G_{f_i}(f_i(\bar{x}))}{\alpha_i(x_0, \bar{x})} \geq C_{(x_0, \bar{x})}\left(\nabla(G_{f_i}(f_i))(\bar{x})\right) + \rho_i \frac{d_i(x_0, \bar{x})}{\alpha_i(x_0, \bar{x})}, \tag{10}$$

$$\frac{G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x}))}{\beta_j(x_0, \bar{x})} \geq C_{(x_0, \bar{x})}\left(\nabla(G_{g_j}(g_j))(\bar{x})\right) + \rho'_j \frac{d'_j(x_0, \bar{x})}{\beta_j(x_0, \bar{x})}, \tag{11}$$

$$\frac{G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))}{\gamma_t(x_0, \bar{x})} \geq C_{(x_0, \bar{x})}\left(\nabla(G_{h_t}(h_t))(\bar{x})\right) + \bar{\rho}_t \frac{\bar{d}_t(x_0, \bar{x})}{\gamma_t(x_0, \bar{x})}, \tag{12}$$

$$\frac{G_{h_t}(h_t(\bar{x})) - G_{h_t}(h_t(x_0))}{\gamma_t(x_0, \bar{x})} \geq C_{(x_0, \bar{x})}\left(-\nabla(G_{h_t}(h_t))(\bar{x})\right) + \bar{\rho}_t \frac{\bar{d}_t(x_0, \bar{x})}{\gamma_t(x_0, \bar{x})}, \tag{13}$$

for $i \in K$, $j \in J(\bar{x})$, $t \in P^+(\bar{x})$ and $t \in P^-(\bar{x})$, respectively. Employing (10),(11), 12) and (13) to (9), we have

$$\begin{aligned}
&\sum_{i=1}^{k} \bar{\lambda}_i C_{(x_0, \bar{x})}\left(\nabla(G_{f_i}(f_i))(\bar{x})\right) + \sum_{j \in J(\bar{x})} \bar{\xi}_j C_{(x_0, \bar{x})}\left(\nabla(G_{g_j}(g_j))(\bar{x})\right) + \\
&\sum_{t \in P^+} \bar{\mu}_t C_{(x_0, \bar{x})}\left(\nabla(G_{h_t}(h_t))(\bar{x})\right) + \sum_{t \in P^-} (-\bar{\mu}_t) C_{(x_0, \bar{x})}\left(-\nabla(G_{h_t}(h_t))(\bar{x})\right) + \\
&\sum_{i=1}^{k} \bar{\lambda}_i \rho_i \frac{d_i(x_0, \bar{x})}{\alpha_i(x_0, \bar{x})} + \sum_{j \in J(\bar{x})} \bar{\xi}_j \rho'_j \frac{d'_j(x_0, \bar{x})}{\beta_j(x_0, \bar{x})} + \sum_{t \in P^{+-}} |\bar{\mu}_t| \bar{\rho}_t \frac{\bar{d}_t(x_0, \bar{x})}{\gamma_t(x_0, \bar{x}))} < 0
\end{aligned} \tag{14}$$

By (7) and the convexity of C, we can conclude that

$$C_{(x_0, \bar{x})}\left(\frac{1}{\delta}\left[\sum_{i=1}^{k} \bar{\lambda}_i \nabla(G_{f_i}(f_i))(\bar{x}) + \sum_{j \in J(\bar{x})} \bar{\xi}_j \nabla(G_{g_j}(g_j))(\bar{x}) + \sum_{t \in P^{+-}} \bar{\mu}_t \nabla(G_{h_t}(h_t))(\bar{x})\right]\right) < 0$$

where $\delta = \sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j \in J(\bar{x})} \bar{\xi}_j + \sum_{t \in P^{+-}} |\bar{\mu}_t| > 0$. Note that $\bar{\xi}_j = 0$, $j \in M \backslash J(\bar{x})$ and $\mu_t = 0, t \in P \backslash P^{+-}$, we have a contradiction to (1). Hence, $\bar{x}$ is a weakly efficient solution for (CVP), and the proof is complete. □

Similar to the proof of Theorem 3.2, we can establish Theorems 3.3 and 3.4. Therefore, we simply state them here.

Theorem 3.3 (G-KKT sufficient optimality condition)**.** *Let $\bar{x}$ be a feasible point for (CVP) such that conditions (1)-(3) are satisfied at $\bar{x}$. Assume that f is strictly $(G_f, C, \alpha, \rho, d)$-convex at $\bar{x}$, $g_{J(\bar{x})}$ is $(G_{g_{J(\bar{x})}}, C, \beta, \rho', d')$-convex at $\bar{x}$ on X, $h_t\,(t \in P^+(\bar{x}))$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-convex at $\bar{x}$, and $h_t\,(t \in P^-(\bar{x}))$ is*

$(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-*concave at* $\bar{x}$ *on* X, *where* $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ *and* $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. *Then* $\bar{x}$ *is an efficient solution for (CVP).*

Theorem 3.4 (G-KKT sufficient optimality condition)**.** *Let* $\bar{x}$ *be a feasible point for (CVP) such that conditions (1)-(3) are satisfied at* $\bar{x}$. *Assume that* f *is* $(G_f, C, \alpha, \rho, d)$-*convex at* $\bar{x}$, $g_{J(\bar{x})}$ *is* $(G_{g_{J(\bar{x})}}, C, \beta, \rho', d')$-*convex at* $\bar{x}$, $h_t\,(t \in P^+(\bar{x}))$ *is* $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-*convex at* $\bar{x}$, *and* $h_t\,(t \in P^-(\bar{x}))$ *is* $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-*concave at* $\bar{x}$ *on* X, *where* $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ *and* $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. *If the following inequality*

$$\sum_{i=1}^{k} \bar{\lambda}_i \rho_i \frac{d_i(x, \bar{x})}{\alpha_i(x, \bar{x})} + \sum_{j=1}^{m} \bar{\xi}_j \rho'_j \frac{d'_j(x, \bar{x})}{\beta_j(x, \bar{x})} + \sum_{t=1}^{p} |\bar{\mu}_t| \bar{\rho}_t \frac{\bar{d}_t(x, \bar{x})}{\gamma_t(x, \bar{x})} > 0$$

holds for any $x \neq \bar{x} \in X$. *Then* $\bar{x}$ *is an efficient solution for (CVP).*

4 G-Mond-Weir vector duality

Making use of the optimality conditions of the preceding section, we consider the following multiobjective dual problem in relation to (CVP), which is in the format of Mond-Weir [9]. And, we call it the G-Mond-Weir vector dual problem for the multiobjective programming problem (CVP).

$$\begin{aligned}
(GMWD)\ \ & \max f(y) := (f_1(y), f_2(y), \cdots, f_k(y)), \\
& s.t. \sum_{i=1}^{k} \lambda_i \nabla (G_{f_i}(f_i))(y) + \sum_{j=1}^{m} \xi_j \nabla (G_{g_j}(g_j))(y) \\
& \qquad + \sum_{t=1}^{p} \mu_t \nabla (G_{h_t}(h_t))(y) = 0 \qquad (15) \\
& \xi^T (G_g(g(y)) - G_g(0)) \geq 0 \\
& \mu^T (G_h(h(y)) - G_h(0)) = 0 \\
& \lambda \in \mathbb{R}^n, \lambda \geqslant 0, \lambda^T \mathbf{e} = 1, \\
& \xi \in \mathbb{R}^m, \xi \geqq 0, \mu \in \mathbb{R}^p, y \in X
\end{aligned}$$

where $\mathbf{e} = (1, \cdots, 1)^T$, G_{f_i}, $i \in K$, are strictly increasing functions defined on $I_{f_i}(X)$, $G_{g_j}, j \in M$, are strictly increasing functions defined on $I_{g_j}(X)$, and $G_{h_t}, t \in P$, are strictly increasing functions defined on $I_{h_t}(X)$, such that $G_f(f)$, $G_g(g)$ and $G_h(h))$ are differential on X.

Let W denote the set of all feasible points of $(GMWD)$ and $pr_X W$ be the projection of the set W on X, that is, $pr_X W := \{y \in X : (y, \lambda, \xi, \mu) \in W\}$. Further, for a given $(y, \lambda, \xi, \mu) \in W$, we denote by $P^+(y)$ and $P^-(y)$ (or simply P^+ and P^-) the sets of equality constraints indices for which the corresponding Lagrange multiplier is positive and negative, respectively, that is, $P^+(y) = \{t \in P : \mu_t > 0\}$ and $P^-(y) = \{t \in P : \mu_t < 0\}$.

Theorem 4.1 (G-Weak duality)**.** *Let x and (y, λ, ξ, μ) be (CVP)-feasible and (GMWD)-feasible, respectively. Assume that f is $(G_f, C, \alpha, \rho, d)$-convex at y on $X \cup pr_X W$, g is $(G_g, C, \beta, \rho', d')$-convex at y on $X \cup pr_X W$, $h_t\,(t \in P^+)$ is $(G_{h_t}, C,\ \gamma_t, \bar{\rho}_t, \bar{d}_t)$-convex at y on $X \cup pr_X W$, and $h_t\,(t \in P^-)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-concave at y on $X \cup pr_X W$. Moreover the inequality*

$$\sum_{i=1}^{k} \lambda_i \rho_i \frac{d_i(x,y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} \xi_j \rho'_j \frac{d'_j(x,y)}{\beta_j(x,y)} + \sum_{t=1}^{p} |\mu_t| \bar{\rho}_t \frac{\bar{d}_t(x,y)}{\gamma_t(x,y)} \geqslant 0 \tag{16}$$

holds, then $f(x) \not< f(y)$.

Proof. Suppose to the contrary that $f(x) \not< f(y)$. Therefore, we obtain

$$f(x) < f(y).$$

or

$$G_{f_i}(f_i(x)) < G_{f_i}((f_i(y)), i = 1, \ldots, k.$$

Note that

$$\begin{aligned} &g(x) \leqq 0,\ \xi^T(G_g(g(y)) - G_g(0)) \geq 0,\ \xi \geqq 0, \\ &h(x) = 0, \mu^T(G_h(h(y)) - G_h(0)) = 0, \mu \in \mathbb{R}^p. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i \frac{G_{f_i}(f_i(x)) - G_{f_i}(f_i(y))}{\alpha_i(x,y)} &+ \sum_{j=1}^{m} \xi_j \frac{G_{g_j}(g_j(x)) - G_{g_j}(g_j(y))}{\beta_j(x,y)} \\ &+ \sum_{t \in P^{+-}} |\mu_t| \frac{G_{h_t}(h_t(x)) - G_{h_t}(h_t(y))}{\gamma_t(x,y)} < 0 \end{aligned} \tag{17}$$

where $P^{+-} = P^+ \cup P^-$. By the generalized invexity assumption of f, g and h, we have

$$\frac{G_{f_i}(f_i(x)) - G_{f_i}(f_i(y))}{\alpha_i(x,y)} \geq C_{(x,y)}\left(\nabla(G_{f_i}(f_i))(y)\right) + \rho_i \frac{d_i(x,\,y)}{\alpha_i(x,\,y)}, \tag{18}$$

$$\frac{G_{g_j}(g_j(x)) - G_{g_j}(g_j(y))}{\beta_j(x,\,y)} \geq C_{(x,y)}\left(\nabla(G_{g_j}(g_j))(y)\right) + \rho'_j \frac{d'_j(x,\,y)}{\beta_j(x,\,y)}, \tag{19}$$

$$\frac{G_{h_t}(h_t(x)) - G_{h_t}(h_t(y))}{\gamma_t(x,\,y)} \geq C_{(x,y)}\left(\nabla(G_{h_t}(h_t))(y)\right) + \bar{\rho}_t \frac{\bar{d}_t(x,\,y)}{\gamma_t(x,y)}, \tag{20}$$

$$\frac{G_{h_t}(h_t(y)) - G_{h_t}(h_t(x))}{\gamma_t(x,\,y)} \geq C_{(x,y)}\left(-\nabla(G_{h_t}(h_t))(y)\right) + \bar{\rho}_t \frac{\bar{d}_t(x,\,y)}{\gamma_t(x,y)}, \tag{21}$$

for $i \in K$, $j \in M$, $t \in P^+(y)$ and $t \in P^-(y)$, respectively. Employing (18),(19), 20) and (21) to (17), we have

$$\sum_{i=1}^{k} \lambda_i C_{(x,y)} \left(\nabla (G_{f_i}(f_i))(y)\right) + \sum_{j \in J(y)} \xi_j C_{(x,y)} \left(\nabla (G_{g_j}(g_j))(y)\right) +$$
$$\sum_{t \in P^+} \mu_t C_{(x,y)} \left(\nabla (G_{h_t}(h_t))(y)\right) + \sum_{t \in P^-} (-\mu_t) C_{(x,y)} \left(-\nabla (G_{h_t}(h_t))(y)\right) +$$
$$\sum_{i=1}^{k} \lambda_i \rho_i \frac{d_i(x,\, y)}{\alpha_i(x,\, y)} + \sum_{j \in M} \xi_j \rho'_j \frac{d'_j(x,\, y)}{\beta_j(x,\, y)} + \sum_{t \in P^{+-}} |\mu_t| \bar{\rho}_t \frac{\bar{d}_t(x,\, y)}{\gamma_t(x, y))} < 0 \tag{22}$$

By (23), the convexity of C and $\mu_t = 0, t \in P \backslash P^{+-}$, we can conclude that

$$C_{(x,y)} \left(\frac{1}{\delta} \left[\sum_{i=1}^{k} \lambda_i \nabla (G_{f_i}(f_i))(y) + \sum_{j=1}^{m} \xi_j \nabla (G_{g_j}(g_j))(y) + \sum_{t=1}^{p} \mu_t \nabla (G_{h_t}(h_t))(y) \right] \right) < 0$$

where $\delta = 1 + \sum_{j=1}^{m} \xi_j + \sum_{t=1}^{p} |\mu_t| > 0$. Hence, we have a contradiction to (15). □

Theorem 4.2 (G-Weak duality). *Let x and (y, λ, ξ, μ) be (CVP)-feasible and (GMWD)-feasible, respectively. Assume that f is strictly $(G_f, C, \alpha, \rho, d)$-convex at y on $X \cup pr_X W$, g is $(G_g, C, \beta, \rho', d')$-convex at y on $X \cup pr_X W$, $h_t\,(t \in P^+)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-convex at y on $X \cup pr_X W$, and $h_t\,(t \in P^-)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-concave at y on $X \cup pr_X W$. Moreover the inequality*

$$\sum_{i=1}^{k} \lambda_i \rho_i \frac{d_i(x,y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} \xi_j \rho'_j \frac{d'_j(x,y)}{\beta_j(x,y)} + \sum_{t=1}^{p} |\mu_t| \bar{\rho}_t \frac{\bar{d}_t(x,y)}{\gamma_t(x,y)} \geqslant 0 \tag{23}$$

holds, then $f(x) \nleq f(y)$.

Theorem 4.3 (G-Strong duality). *Let $\bar{x}$ be a (weak) efficient solution of (CVP) and the hypothesis of Theorem 3.1 holds. Then there exist $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ is a $(GMWD)$-feasible point. If also the hypothesis of Theorem 4.1 holds for all $(GMWD)$-feasible points (y, λ, ξ, μ), then $(\bar{x}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ is a (weak) maximum for $(GMWD)$, and the objective functions values are equal in problems (CVP) and $(GMWD)$.*

Proof. By Theorem 3.1, there exist $\bar{\lambda} \in \mathbb{R}^k$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$, such that the G-Karush-Kuhn-Tucker necessary optimality conditions (1)-(3) hold. From

(2), (3), $g(x) \leqq 0$ and $h(\bar{x}) = 0$, we have

$$\sum_{j=1}^{m} \bar{\xi}_j \left(G_{g_j}(g_j(\bar{x})) - G_{g_j}(0)\right) \geq \sum_{j=1}^{m} \bar{\xi}_j \left(G_{g_j}(g_j(\bar{x})) - G_{g_j}(g_j(x))\right) \geq 0 \tag{24}$$

$$\sum_{t=1}^{p} \bar{\mu}_t \left(G_{h_t}(h_t(\bar{x})) - G_{h_t}(0)\right) = 0. \tag{25}$$

Hence $(\bar{x}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ is a $(GMWD)$ feasible solution from (1), (24) and (25). Also, by G-weak duality (Theorem 4.1 or 4.2), it follows that $(\bar{x}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ is a (weak) maximum in $(GMWD)$. □

Theorem 4.4 (G-converse duality)**.** *Let $(\bar{y}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ be a (weak) maximum for $(GMWD)$ and $\bar{y} \in X$. Assume that f is (strictly) $(G_f, C, \alpha, \rho, d)$-convex at $\bar{y}$ on $X \cup pr_X W$, g is $(G_g, C, \beta, \rho', d')$-convex at $\bar{y}$ on $X \cup pr_X W$, h_t $(t \in P^+)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-convex at $\bar{y}$ on $X \cup pr_X W$, and h_t $(t \in P^-)$ is $(G_{h_t}, C, \gamma_t, \bar{\rho}_t, \bar{d}_t)$-concave at $\bar{y}$ on $X \cup pr_X W$. Moreover, the inequality*

$$\sum_{i=1}^{k} \bar{\lambda}_i \rho_i \frac{d_i(x, \bar{y})}{\alpha_i(x, \bar{y})} + \sum_{j=1}^{m} \bar{\xi}_j \rho'_j \frac{d'_j(x, \bar{y})}{\beta_j(x, \bar{y})} + \sum_{t=1}^{p} |\bar{\mu}_t| \bar{\rho}_t \frac{\bar{d}_t(x, \bar{y})}{\gamma_t(x, \bar{y})} \geqslant 0 \tag{26}$$

holds for all $x \in X$. Then $\bar{y}$ is a (weak) efficient solution in (CVP)

Proof. Let $(\bar{y}, \bar{\lambda}, \bar{\xi}, \bar{\mu})$ be a (weak) maximum for $(GMWD)$ and $\bar{y} \in X$. Therefore

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla (G_{f_i}(f_i))(\bar{y}) + \sum_{j=1}^{m} \bar{\xi}_j \nabla (G_{g_j}(g_j))(\bar{y}) + \sum_{t=1}^{p} \bar{\mu}_t \nabla (G_{h_t}(h_t))(\bar{y}) = 0 \tag{27}$$

Suppose to the contrary that $\bar{y}$ is not a (weak) efficient solution in (CVP). Then, there exists $x_0 \in X$ such that

$$f(x_0) \leqslant (<) f(\bar{y})$$

Note that

$$g(x_0) \leqq 0,\ \bar{\xi}^T (G_g(g(\bar{y})) - G_g(0)) \geq 0,\ \xi \geqq 0,$$
$$h(x_0) = 0, \bar{\mu}^T (G_h(h(\bar{y})) - G_h(0)) = 0, \bar{\mu} \in \mathbb{R}^p.$$

we have the inequality

$$\sum_{i=1}^{k} \bar{\lambda}_i \frac{G_{f_i}(f_i(x_0)) - G_{f_i}(f_i(\bar{y}))}{\alpha_i(x_0, \bar{y})} + \sum_{j=1}^{m} \bar{\xi}_j \frac{G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{y}))}{\beta_j(x_0, \bar{y})}$$
$$+ \sum_{t \in P^{+-}} |\bar{\mu}_t| \frac{G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{y}))}{\gamma_t(x_0, \bar{y})} < 0 \tag{28}$$

where $P^{+-} = P^{+} \cup P^{-}$. Using similar arguments as in the proof of Theorem 4.1, we have

$$
\begin{aligned}
&C_{(x_0,\bar{y})}\left(\frac{1}{\delta}\left(\sum_{i=1}^{k}\bar{\lambda}_i\nabla(G_{f_i}(f_i))(\bar{y})+\sum_{j=1}^{m}\bar{\xi}_j\nabla(G_{g_j}(g_j))(\bar{y})\right.\right.\\
&\left.\left.+\sum_{t=1}^{p}\bar{\mu}_t\nabla(G_{h_t}(h_t))(\bar{y})\right)\right)\leqslant\frac{1}{\delta}\left(\sum_{i=1}^{k}\bar{\lambda}_i\frac{G_{f_i}(f_i(x_0))-G_{f_i}(f_i(\bar{y}))}{\alpha_i(x_0,\bar{y})}\right.\\
&\left.+\sum_{j=1}^{m}\bar{\xi}_j\frac{G_{g_j}(g_j(x_0))-G_{g_j}(g_j(\bar{y}))}{\beta_j(x_0,\bar{y})}+\sum_{t\in P^{+-}}|\bar{\mu}_t|\frac{G_{h_t}(h_t(x_0))-G_{h_t}(h_t(\bar{y}))}{\gamma_t(x_0,\bar{y})}\right)\\
&-\frac{1}{\delta}\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i\frac{d_i(x_0,\bar{y})}{\alpha_i(x_0,\bar{y})}+\sum_{j=1}^{m}\bar{\xi}_j\rho'_j\frac{d'_j(x_0,\bar{y})}{\beta_j(x_0,\bar{y})}+\sum_{t=1}^{p}|\bar{\mu}_t|\bar{\rho}_t\frac{\bar{d}_t(x_0,\bar{y})}{\gamma_t(x_0,\bar{y})}\right) \qquad (29)
\end{aligned}
$$

where $\delta = 1 + \sum_{j=1}^{m}\bar{\xi}_j + \sum_{t=1}^{p}|\bar{\mu}|_t$. Combining (26), (27)and (29), we deduce

$$
\begin{aligned}
&\sum_{i=1}^{k}\bar{\lambda}_i\frac{G_{f_i}(f_i(x_0))-G_{f_i}(f_i(\bar{y}))}{\alpha_i(x_0,\bar{y})}+\sum_{j=1}^{m}\bar{\xi}_j\frac{G_{g_j}(g_j(x_0))-G_{g_j}(g_j(\bar{y}))}{\beta_j(x_0,\bar{y})}\\
&\quad+\sum_{t\in P^{+-}}|\bar{\mu}_t|\frac{G_{h_t}(h_t(x_0))-G_{h_t}(h_t(\bar{y}))}{\gamma_t(x_0,\bar{y})}\geqslant 0.
\end{aligned}
$$

This contradicts (28). □

5 Conclusions

In this paper, we consider a class of multiobjective programming problems in which each component of the objective function is not necessarily differential. We have proved new G-necessary optimality conditions for multiobjective programming problems with both inequality and equality constraints involving the functions which are not necessarily differential. To relax convexity assumptions imposed on theorems on sufficient optimality conditions and duality, we present a new generalized convexity, which is called (G, C, α, ρ, d)-convexity and which is an extension including (C, α, ρ, d)-convexity[20, 7], (F, α, ρ, d)-convexity[14, 15] and G-invexity [2, 3, 4]. Basing on this new generalized convexity, we establish new G-sufficient optimality conditions for the multiobjective programming problem. Further, a general G-Mond-Weirdual problem is introduced for the considered multiobjective programming problem. Our results extend and improve the corresponding results in the literature.

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