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Power Allocation over Nakagami-Lognormal Fading Channels with Outage Probability Specifications

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Abstract

In this paper, a conservative but tractable approximation method for outage-based power allocation over composite lognormal shadowing Nakagami fading channels (with independent interference) is proposed.

Keywords: Nakagami-m distribution, shadowed fading channels, power control, outage probability, chance-constrained programming, geometric programming (GP)

1 Introduction

In wireless communication systems, an important quality of service (QoS) requirement is that the outage probability of each transmitter/receiver pair is kept below a given level [9]. The problem of minimizing power consumption with these outage probability specifications over wireless fading channels can be posed as a stochastic program. In general, it is challenging to solve this stochastic program exactly, except for the special case of Rayleigh fading channels [5].

In this paper, we consider the problem of power allocation with outage probability specifications, over composite lognormal shadowing Nakagami fading channels (with independent interference). We formulate this problem as a specific type of (chance-constrained) stochastic programming problem, called the *chance-constrained geometric program (CCGP)* [4]. In general it is very hard to solve this chance-constrained GP; instead, a conservative approximation, which is a (deterministic) GP, is solved to find a suboptimal solution of the CCGP. A numerical example is given to demonstrate the effectiveness of the proposed approximation scheme.

2 Nakagami-Lognormal Fading Model

2.1 Prerequisites

A (two-parameter) gamma distributed random variable X with the scale parameter $\theta > 0$ and the shape parameter $k > 0$, denoted by $X \sim \mathbf{Gamma}(k, \theta)$, has the probability density function

$$f_X(x) = \frac{x^{k-1}}{\theta^k \Gamma(k)} \exp\left(-\frac{x}{\theta}\right), \quad x > 0,$$

where $\Gamma(\cdot)$ is the gamma function. Note that the scale parameter θ can be represented in terms of the rth moment of X as

$$\theta = (\mathbf{E}(X^r)\Gamma(k)/\Gamma(k+r))^{1/r}, \quad r > 0, \quad r \in \mathbb{Z}.$$

The rth *negative* moment of the gamma random variable X exists only for $r < k$; in this case, it is given by

$$\mathbf{E}(X^{-r}) = \frac{\Gamma(k-r)}{\theta^r \Gamma(k)}, \quad k > r > 0, \quad r \in \mathbb{Z}. \tag{1}$$

(See, *e.g.*, [1].)

The family of Nakagami (or Nakagami-m) distributions [7] has a shape parameter $m \geq 1/2$ and a scale parameter $\Omega > 0$ controlling the distribution spread. The Nakagami distribution is related to the gamma distribution in the way that, given $X \sim \mathbf{Gamma}(k, \theta)$,

$$X^{1/2} \sim \mathbf{Nakagami}(m, \Omega), \quad m = k, \quad \Omega = k\theta = \mathbf{E}\, X.$$

Let $\mathcal{LN}\,(\mu, \sigma^2)$ denote the (two-parameter) lognormal distribution. Given $V \sim \mathcal{LN}\,(\mu, \sigma^2)$, the arbitrary power of V is still lognormally distributed and its expectation is given by

$$\mathbf{E}\, V^r = \exp\left(r\mu + r^2\sigma^2/2\right), \quad r \in \mathbb{R}. \tag{2}$$

2.2 The model

In this paper the following setting for shadowed fading channels is considered. We have n transmitters, labelled $1, \ldots, n$, which transmit at (positive) power levels $P_1, \ldots, P_n$ respectively. We also have n receivers, labelled $1, \ldots, n$; receiver i is meant to receive the signal from transmitter i, *i.e.*, the channel between transmitter i and receiver j is undesired (interfering) if $i \neq j$. The (i, j)th channel refers to the channel from the transmitter j to the receiver i. At the receiver i the power received from transmitter, denoted by R_{ij}, is given by

$$R_{ij} = G_{ij} C_{ij} P_j. \tag{3}$$

Here the components G_{ij}, which is positive, represents the path gain (not including fading) of the (i, j)th channel. We assume that G_{ij} are *constant*, *i.e.*, they do not change (significantly) over each power interval. In addition, we also assume that C_{ij} are *independent*.

In a Nakagami-lognormal fading environment, the distribution of the compound shadowing-fading factor C_{ij} in (3), due to Nakagami fading with lognormal shadowing, can be represented as

$$C_{ij} \sim \mathbf{Gamma}\,(m_{ij}, \Omega_{ij}/m_{ij}) \bigwedge_{\Omega_{ij}} \mathcal{LN}(\mu_{ij}, \sigma_{ij}^2), \tag{4}$$

that is,

$$C_{ij}|S_{ij} \sim \mathbf{Gamma}\,(m_{ij}, S_{ij}/m_{ij})\,, \quad S_{ij} \sim \mathcal{LN}(\mu_{ij}, \sigma_{ij}^2), \tag{5}$$

or equivalently,

$$C_{ij}^{1/2} \sim \mathbf{Nakagami}\,(m_{ij}, \Omega_{ij}) \bigwedge_{\Omega_{ij}} \mathcal{LN}(\mu_{ij}, \sigma_{ij}^2),$$

where m_{ij} is the Nakagami shape factor (or called the fading severity index) of the (i, j)th channel, and $\Omega_{ij} = \mathbf{E}(C_{ij}|S_{ij} = \Omega_{ij})$; see, *e.g.*, [2]. (Here the "mixing" operator $f \bigwedge_{\alpha} g$ mixes the distributions f and g and returns a (compound) distribution which is distributed as f with the quantity α (of f) being a random variable distributed as g.) Note that $C_{ij}^{1/2}$ and C_{ij} represent the fading "envelope" and "squared-envelope" of the (i, j)th channel respectively.

It follows from the law of iterated expectations that, for positive integers r satisfying $m_{ij} > r$, the rth negative moment of the gamma-lognormal mixture C_{ij} exists and is given by

$$\begin{aligned}
\mathbf{E}\, C_{ij}^{-r} &= \mathbf{E}(\mathbf{E}(C_{ij}^{-r}|S_{ij})) \\
&= \mathbf{E}\left(\left(\frac{m_{ij}}{S_{ij}}\right)^r \frac{\Gamma(m_{ij}-r)}{\Gamma(m_{ij})}\right) \\
&= \frac{m_{ij}^r \Gamma(m_{ij}-r)}{\Gamma(m_{ij})}\, \mathbf{E}\, S_{ij}^{-r} \\
&= \frac{m_{ij}^r \Gamma(m_{ij}-r)}{\Gamma(m_{ij})} \exp\left(\frac{1}{2} r^2 \sigma_{ij}^2 - r\mu_{ij}\right). \qquad (6)
\end{aligned}$$

(Here (1) and (2) are applied.) The rth positive moment of C_{ij} can be derived in the same way:

$$\mathbf{E}\, C_{ij}^r = \mathbf{E}\left(\left(\frac{S_{ij}}{m_{ij}}\right)^r \frac{\Gamma(m_{ij}+r)}{\Gamma(m_{ij})}\right) = \frac{\Gamma(m_{ij}+r)}{m_{ij}^r \Gamma(m_{ij})} \exp\left(\frac{1}{2} r^2 \sigma_{ij}^2 + r\mu_{ij}\right). \qquad (7)$$

Obviously, since C_{ij} are positively distributed, (positive and negative) moments of C_{ij}, if exist, are positive as well.

3 Power Allocation via Geometric Programming

Let $\mathbb{R}_{++}$ denote the set of positive real numbers. A real valued function of the form

$$f(x) = \sum_{k=1}^{K} c_k \prod_{j=1}^{n} x_j^{a_{jk}},$$

where $c_k > 0$ and $a_{jk} \in \mathbb{R}$, is called a *posynomial.* An optimization problem of the form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 1, \quad i = 1, \ldots, m, \end{array}$$

where $x \in \mathbb{R}_{++}^n$ is the optimization variable and $f_0, f_1, \ldots, f_m$ are posynomials, is called a *(posynomial) geometric program (GP).* GPs (in posynomial form) can be reformulated as convex problems and hence can be globally and efficiently solved. (For more on GP, see the tutorial [3] and references therein.)

3.1 Outage-based power minimization

In the fading channel model (3), the signal power at the sth receiver is $G_{ss}C_{ss}P_s$, subject to the total interference power $\sum_{k\neq s} G_{sk}C_{sk}P_k$. For the sth receiver/transmitter pair, its *noise-plus-interference-to-signal ratio (NISR)* is defined as the reciprocal of its SINR, *i.e.*,

$$\begin{aligned} N_s(P_i; C_{ij}) &= \left(\frac{G_{ss}C_{ss}P_s}{\beta_s + \sum_{k\neq s} G_{sk}C_{sk}P_k} \right)^{-1} \\ &= \beta_s G_{ss}^{-1} C_{ss}^{-1} P_s^{-1} + \sum_{k\neq s} G_{sk} G_{ss}^{-1} C_{sk} C_{ss}^{-1} P_k P_s^{-1}, \end{aligned} \tag{8}$$

where $\beta_s \geq 0$ represents the noise power at the sth receiver, assumed to be constant. (For convenience, we will use the newly defined NISR, instead of the commonly used SINR, as the measure comparing the level of a desired signal with the level of interference power plus noise.)

One common QoS requirement is that, for each receiver/transmitter pair, the SINR must be kept above a given threshold $(N_s^{\max})^{-1}$. This QoS requirement can be represented as

$$N_s \leq N_s^{\max}, \quad s = 1, \dots, n.$$

The *outage probability* of the sth receiver/transmitter pair is now given by

$$\mathcal{O}_s(P_1, \dots, P_n) = \mathbf{Prob}\left(N_s > N_s{}^{\max}\right), \quad s = 1, \dots, n. \tag{9}$$

Here, $\mathbf{Prob}(A)$ is the probability of event A. The outage probability can be interpreted as the fraction of time that the sth transmitter/receiver pair experiences an outage due to fading. In our expression for the outage probability $\mathcal{O}_s$, statistical variations of both received signal power and received interference power are taken into account.

The QoS requirement we consider in this paper is that the outage probability (of the sth receiver/transmitter pair) must be kept below a given threshold $o_s^{\max}$:

$$\mathcal{O}_s(P_1, \dots, P_n) \leq o_s^{\max}, \quad s = 1, \dots, n.$$

The problem of finding the minimum power allocation with the above outage probability constraints can be formulated as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n P_i \\ \text{subject to} & \mathcal{O}_s(P_1, \dots, P_n) \leq o_s^{\max}, \quad s = 1, \dots, n. \end{array} \tag{10}$$

In contrast to the deterministic power control problem (see, *e.g.*, [8]), the stochastic power control problem (10) appears to be challenging to solve exactly. Even evaluating the outage probability of individual channel is already difficult, especially for the channels with compound fading and shadowing (for example, the composite Nakagami-lognormal channels we consider in this paper). As an important special case, for interference-limited wireless networks with Rayleigh fading (in both the desired and interference signals), this problem can be cast as a (deterministic) geometric program and hence can be solved exactly; the reader is referred to [5] for details.

3.2 Conservative approximation via GP

In the following we describe an outage-based power allocation method for Nakagami-lognormal channels. In §2.2, we have assumed that the compound shadowing-fading components C_{ij} in (3) are independent. Note that the NISR (8) has the form of a posynomial in $P_1, \ldots, P_n$, with *random* coefficients. Then the stochastic minimization problem (10), rewritten as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n P_i \\ \text{subject to} & \mathbf{Prob}\left((N_s^{\max})^{-1} N_s(P_i; C_{ij}) > 1\right) \le o_s^{\max}, \quad s = 1, \ldots, n, \end{array} \tag{11}$$

is inherently a chance-constrained GP. Applying the approximation procedure illustrated in [4] to this CCGP, we can obtain a *conservative* approximation, *i.e.*, the so-called *Cantelli approximation*

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n P_i \\ \text{subject to} & (N_s^{\max})^{-1}\, \mathbf{E}\, N_s(P_i; C_{ij}) + \gamma_s t_s^{1/2} \le 1, \quad s = 1, \ldots, n, \\ & t_s^{-1} (N_s^{\max})^{-2}\, \mathbf{Var}\, N_s(P_i; C_{ij}) \le 1, \quad s = 1, \ldots, n, \end{array} \tag{12}$$

where $P_i > 0$, $t_i > 0$, $i = 1, \ldots, n$ are optimization variables, and

$$\gamma_s = \sqrt{1/o_s^{\max} - 1},$$

$$\mathbf{E}\, N_s(P_i; C_{ij}) = \beta_s G_{ss}^{-1}\, \mathbf{E}\, C_{ss}^{-1} P_s^{-1} + \sum_{k \neq s} G_{sk} G_{ss}^{-1}\, \mathbf{E}\, C_{sk}\, \mathbf{E}\, C_{ss}^{-1} P_k P_s^{-1}, \tag{13}$$

$$\begin{aligned} \mathbf{Var}\, N_s(P_i; C_{ij}) &= \beta_s^2 G_{ss}^{-2}\, \mathbf{Var}\, C_{ss}^{-1} P_s^{-2} + \sum_{k \neq s} G_{sk}^2 G_{ss}^{-2}\, \mathbf{Var}(C_{sk} C_{ss}^{-1}) P_k^2 P_s^{-2} \\ &+ 2 \sum_{k \neq s} \beta_s G_{sk} G_{ss}^{-2}\, \mathbf{Cov}(C_{ss}^{-1}, C_{sk} C_{ss}^{-1}) P_k P_s^{-2} \\ &+ 2 \sum_{\substack{1 \le k < \ell \le n \\ k \neq s,\ \ell \neq s}} G_{sk} G_{s\ell} C_{ss}^{-2}\, \mathbf{Cov}(C_{sk} C_{ss}^{-1}, C_{s\ell} C_{ss}^{-1}) P_k P_\ell P_s^{-2}. \end{aligned} \tag{14}$$

It is a simple exercise to show that the values of variance and covariance terms here can be computed with (6) and (7), using

$$\mathbf{Var}\, C_{ss}^{-1} = \mathbf{E}\, C_{ss}^{-2} - (\mathbf{E}\, C_{ss}^{-1})^2, \tag{15}$$

$$\mathbf{Var}(C_{sk} C_{ss}^{-1}) = \mathbf{E}\, C_{sk}^2\, \mathbf{E}\, C_{ss}^{-2} - (\mathbf{E}\, C_{sk})^2 (\mathbf{E}\, C_{ss}^{-1})^2, \tag{16}$$

$$\mathbf{Cov}(C_{ss}^{-1}, C_{sk} C_{ss}^{-1}) = \mathbf{E}\, C_{sk}\, \mathbf{Var}\, C_{ss}^{-1}, \tag{17}$$

$$\mathbf{Cov}(C_{sk} C_{ss}^{-1}, C_{s\ell} C_{ss}^{-1}) = \mathbf{E}\, C_{sk}\, \mathbf{E}\, C_{s\ell}\, \mathbf{Var}\, C_{ss}^{-1}. \tag{18}$$

It follows from (13) that $\mathbf{E}\, N_s$ (if exist) are posynomials in $P_1, \ldots, P_n$. In addition, since the covariance terms in (14) are positive, $\mathbf{Var}\, N_s$ (if exist) are also posynomials in $P_1, \dots, P_n$. By the way, although the approximation (12) is conservative (in the sense that its feasible set is a subset of that of (11)), it is a (deterministic) geometric program and hence can be solved globally.

To ensure the existence of $\mathbf{E}\, N_s$ and $\mathbf{Var}\, N_s$ (for bounded P_i), we technically assume that, for all the *desired* receiver/transmitter pairs, the first two negative moments of the gamma-lognormal mixture C_{ii} exist or, equivalently by (6), the condition

$$m_{ii} > 2, \quad i = 1, \dots, n \tag{19}$$

holds. (In this cases, it follows from (15)–(18) that all the variance and covariance terms in (14) exist.) This assumption is mild in general, since we often use the Nakagami distribution to model fading conditions that are more severe than Rayleigh fading (which corresponds to the case with $m_{ij} = 1$); see, *e.g.*, [6]. We conclude that, for Nakagami-lognormal channels satisfying the assumption (19), suboptimal feasible solutions of the stochastic power minimization problem (11) can be obtained by solving the GP (12).

4 Numerical example

In this example we consider a Nakagami-lognormal channel with 50 transmitters and receivers. Each shadowing-fading factor C_{ij} has the compound distribution (4) with $m_{ij} = 5$, and the underlying lognormal shadowing factor S_{ij} in (5) has mean 0 dB and standard deviation 5 dB. We take all the path gains G_{ii} to be one.

We assume that this Nakagami-lognormal channel is *sparse*, in the sense that each desired channel is interfered only by N neighboring interference channels. In this example, we set $N = 5$. For each transmitter/receiver pair, the rule to choose neighboring interferers is trivial and can be illustrated as follows. For example, for the first receiver, the 5 neighboring interferers are the transmitter 49, 50, 2, 3 and 4. For the 25th receiver, the interfering signals come from the transmitter 23, 24, 26, 27 and 28.

We consider the NISR threshold $N_s^{\max} = 0.5$, 0.6 and 0.7 respectively. (They are assumed to be the same for each transmitter/receiver pair.) For each NISR threshold, we vary the outage probability threshold $o_s^{\max}$ (which, again, is assumed to be the same for each transmitter/receiver pair) from 3×10^{-2} to 9×10^{-2}. For each value of $o_s^{\max}$, we generate 50 instances of the power minimization problem (11), in which all the noise powers β_s and cross gains G_{ij} from interfering neighbors are selected as independent random variables uniformly distributed between 10^{-4} and 10^{-5}. (We take the cross gains from non-neighboring interferers to be zero.) Then we compute the total transmitter power $P_1 + \cdots + P_{50}$ for each problem instance using the (conservative) deterministic approximation method proposed in §3.2.

We summarize the numerical results as follows. For a fixed NISR threshold, the total transmitter power increases as lower outage probability thresholds are assigned. In this example, the total transmitter power is sensitive to the NISR threshold when $o_s^{\max}$ is sufficiently small. For instance, with $N_s^{\max} = 0.6$ and 0.7, the trade-off curves between the total transmitter power and the outage probability threshold essentially stay flat (that is, the total power is quite insensitive to the outage probability threshold). However, for a bit lower NISR threshold $N_s^{\max} = 0.5$, the total power increases drastically as $o_s^{\max}$ decreases below 5×10^{-2}. Actually, the corresponding (deterministic) geometric program (12) becomes infeasible quickly for $o_s^{\max}$ below 3×10^{-2}. We conclude that, in this situation, the "price of robustness" is high.

5 Conclusion

In this paper, we have described a power control scheme for Nakagami fading channels with lognormal shadowing. With a good compromise between computational complexity and performance, the proposed conservative approximation method, which is to solve a GP, can efficiently find a power allocation that meets the outage probability specifications.

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