Optimal Reinsurance and Investment Policies under HARA Utility

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Abstract

We study an optimal reinsurance and investment problem under the constant elasticity of variance model for an insurance company. The insurer can buy proportional reinsurance contract and allocate the wealth into two financial securities: a risk-free asset and a risk asset whose price is modeled by a constant elasticity of variance (CEV) model. Assume that the insurer’s cash reserve process is approximated by a Brownian motion with drift, the goal of the insurer is to maximize the expected HARA utility of the terminal wealth. Explicit expression for optimal policies are derived by stochastic control approach and Legendre transform.
Keywords: Constant elasticity of variance; Optimal reinsurance and investment policy; HARA utility; HJB equation; Legendre transform

1 Introduction

Reinsurance and investment decision rules under a continuous-time setting have been a significant topic of actuarial research since the seminal paper Browne (1995). In continuous-times models, the insurer dynamic purchasing reinsurance to transfer the risk to the reinsurer, and investing the finance market consisting of various assets to achieve his management objective. There are two main types of goals for the insurer in the existing literature. The first is minimizing the ruin probability. For more information about this optimization problem, we refer to Azcue and Muler (2013). The second is maximizing the expected utility of his terminal wealth. Many extended studies using this objection (see Yang and Zhang 2005, Gu et al. 2010 and Zhu et al. 2015).

However, the above mentioned researches with the assumptions underlying the Black-Scholes model are often questioned. The geometric Brownian motions (GBMs) cannot describe some observed important empirical features of risk asset price data. In the past three decades, a number of stochastic volatility (SV) models based on more realistic price processes have been proposed. These includes Heston SV model, Constant elasticity of variance (CEV) model and Ornstein-Uhlenbeck (O-U) model. We can refer to Deng et al. (2019) for CEV model; to Zhu et al. (2015), Deng et al. (2018) for Heston model.

In this paper, we study the optimal reinsurance and investment problem for an insurer under CEV model, the insurer aims to maximize the expected Hyperbolic absolute risk aversion (HARA) utility of his terminal wealth. Due to the fact that CARA utility and CRRA utility are all special cases of HARA utility, our paper is a naturally extension of Gu et al. (2010). However, in view of the complexity of HARA utility, it is difficult to solve the nonlinear HJB equation corresponding to optimization problem above by traditional approach straightly. In the spirit of Jung and Kim (2012), we apply the Legendre transform method. First, we obtain the dual transform of the value function by employing the Legendre transform approach. Second, we plug this dual value function into the original nonlinear HJB equation. Consequently, it leads to a new linear HJB equation. Finally, we can derive the explicit expression for the optimal strategy and dual form of the value function.

The rest of the paper is organized as follows. In Section 2, we give formulate the optimization problem for the insurer. In Section 3, we employ the Legendre transform approach to solve the optimization problem. The explicit expression for the optimal strategy is derived.
2 Mathematical descriptions of the problem

Let us consider the complete filtered probability space \((\Omega, \mathcal{F}_t, \mathbb{P})\), where \(\mathcal{F}_t = \sigma(W^0_t, W^1_t, s \leq t)\) is the natural filtration induced by two independent standard Brownian motions \(\{W^0_t\}_{t \geq 0}\) and \(\{W^1_t\}_{t \geq 0}\). The financial market consisting of a risk-free asset and single risky asset. The risk-free asset at time \(t\) by \(B_t\), which evolves according to the following formula:

\[
dB_t = rB_t \, dt, \tag{1}
\]

where \(r > 0\) is a constant rate of interest. The price of the risky asset at time \(t\) by \(P_t\), which is described by the CEV model:

\[
d\frac{P_t}{P_t} = \mu dt + \sigma P^\beta_t \, dW^1_t, \tag{2}
\]

where \(\mu(\mu > r)\) is an expected instantaneous rate of return of the stock. \(\sigma\) and \(\beta\) are constant parameters, and \(\beta\) satisfies the general condition \(\beta < 0\). \(\sigma P^\beta_t\) is the instantaneous volatility.

Here, we set the cumulative claims process \(S_t\) as \(dS_t = adt - bW^0_t\), where \(a\) and \(b\) are positive constants, and satisfying that \(a \gg b\). Assume that the insurer collects premia continuously at the constant rate \(c_0 = (1 + \theta)a\), where \(\theta > 0\) denotes the safety loading. The dynamics of the surplus process \(\{U_t\}_{t \geq 0}\) is given by \(dU_t = c_0 dt - dS_t\). It is straightforward to check that

\[
dU_t = a\theta dt + bW^0_t, \tag{3}
\]

where \(U_0 = u\) denotes the insurer’s initial reserve.

We assume that the insurer purchase proportional reinsurance to reduce the underlying risk involved with her claims process. Let \(q(t) \in [0, 1]\) stand for the reinsurance strategy. Then the premium is paid continuously at the constant rate \(c_1(t) = (1 + \eta)aq(t)\), where \(\eta > \theta\) is the safety loading of the proportional reinsurance. The strategy \((\zeta_t, q_t)\) denotes the strategy followed by the insurer, where \(\zeta_t\) represent the proportion invested in the risky asset. The reserve process subjected to this choice is denoted by \(X_t\), that is

\[
dX_t^\pi = (rX_t + \zeta_t(\mu - r)X_t^\pi + (\theta - \eta q_t)a)dt + \sigma \zeta_t P^\beta_t \, dW^1_t + b(1 - q_t)dW^0_t. \tag{4}
\]

The class of admissible strategies is denoted by \(\mathcal{M}_F\) and is given by

\[
\mathcal{M}_F := \left\{ \pi(s) \in [0, 1] \times \mathbb{R} \text{ is } \mathcal{F}_t \text{ - adapted } \int_t^T \|\pi\|^2 \, ds < \infty \text{ a.s.} \right\}. \tag{5}
\]
2.1 Problem formulate

For the risk reserve process \( \{X^\pi_t\}_{t \geq 0} \) given by (4), that is, we face the problem

\[
V(t, x, p) = \sup_{\pi \in \mathcal{M}} E[U^\pi(X^\pi_t)|(X^\pi_t, P_t) = (x, p)].
\]  

We use the hyperbolic absolute risk aversion (HARA) utility function \( U_{HARA}(x) = U(\iota, \gamma, k; x) \) with parameters \( \iota > 0, \gamma < 1, \gamma \neq 0 \) and \( k > 0 \) defined as follows:

\[
U_{HARA}(x) = \frac{1 - \gamma}{k\gamma} \left( \frac{kx^{1 - \gamma} + \iota}{1 - \gamma} \right)^k.
\]  

Remark 2.1. Note that HARA is a general family of utility functions:

1. Let \( \iota = 0, k = 1 - \gamma \), it leads to the power utility function case;
2. Let \( \iota = 1, \gamma \uparrow -\infty \), then \( \lim_{\gamma \downarrow -\infty} U_{HARA}(x) = U_{CARA} = \frac{1}{k} \exp\{-kx\} \).

3 Solution to HARA utility

3.1 Hamilton-Jacobi-Bellman for the insurer

By using the dynamic programming techniques, then \( V(t, x, p) \) satisfies the following HJB equation

\[
\begin{cases}
\sup_{\pi \in \mathcal{M}} A^\pi V(t, x, p) = 0, \\
V(T, x, p) = U(x),
\end{cases}
\]  

where

\[
A^\pi V(t, x, p) = V_t + (r x + \zeta(\mu - r)x + (\theta - \eta q_t)a)V_x + \mu p V_p + \frac{1}{2} \sigma^2 p^{2\beta + 2} V_{pp} \\
+ \sigma^2 \zeta x p^{2\beta + 1} V_{px} + \frac{1}{2} (b^2 (1 - q_t)^2 + \sigma^2 \zeta^2 x^2 p^{2\beta}) V_{xx} = 0.
\]  

with \( V_t, V_p, V_x, V_{px}, V_{pp} \) and \( V_{xx} \) denote partial derivatives of first and second orders with respect to time, price of risky asset and wealth.

The first order maximizing conditions for the optimal strategies \( q^* \) and \( \zeta^* \) are:

\[
\begin{cases}
\dot{q} = 1 + \frac{a \eta V_x}{b^2 V_{xx}} \\
\dot{\zeta} = \frac{(\mu - r)V_x + \sigma^2 p^{2\beta + 1} V_{px}}{\sigma^2 p^{2\beta} V_{xx}}.
\end{cases}
\]  

Putting this in Eq(8), we obtain a partial differential equation (PDE) for the value function \( V \):

\[
V_t + (r x + (\theta - \eta q_t)a)V_x - \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma^2 p^{2\beta}} + \frac{a^2 \eta^2}{b^2} \right) \frac{V_x^2}{V_{xx}} - p(\mu - r) \frac{V_x V_{px}}{V_{xx}} \\
+ \mu p V_p + \frac{1}{2} \sigma^2 p^{2\beta + 2} V_{pp} - \frac{1}{2} \sigma^2 p^{2\beta + 2} \frac{V_{px}^2}{V_{xx}} = 0.
\]
with \( V(T, x, p) = \mathcal{U}(x) \).

Note that the optimal value \( q^*(t, x, p) \) lies in \([0, 1]\), for all \((t, x, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+\), and that \( \bar{q} < 1 \) since \( V_x(t, x, p) > 0 \) and \( V_{xx}(t, x, p) < 0 \). Therefore if \( \bar{q} \geq 0 \), \( q^*(t, p, x) = \bar{q}(t, x, p) \); otherwise, we simply let \( q^*(t, x, p) = 0 \). All these results lead to the following two lemmas which will be helpful for us to find a solution to (8).

**Lemma 3.1.** Let \( \mathcal{D}_1 := \{(t, x, p) : \bar{q} > 0\} \). Suppose \( V(t, x, p) \) satisfies the properties that \( V_x(t, x, p) > 0 \) and \( V_{xx}(t, x, p) < 0 \) is a solution to

\[
V_t + (rx + (\theta - \eta q_t)a)V_x = \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma^2 p^{2\beta}} + \frac{a^2 \eta^2}{b^2} \right) \frac{V_x^2}{V_{xx}} - p(\mu - r) \frac{V_x V_{px}}{V_{xx}} \\
+ \mu p V_p + \frac{1}{2} \sigma^2 p^{2\beta+2} V_{pp} - \frac{1}{2} \sigma^2 p^{2\beta+2} \frac{V_{px}^2}{V_{xx}} = 0 \tag{12}
\]

with terminal condition

\[
V(T, x, p) = \mathcal{U}(x). \tag{13}
\]

Then \( V(t, p, x) \) satisfies the HJB equation (8) on \( \mathcal{D}_1 \) with boundary condition \( V(T, x, p) = \mathcal{U}(x) \).

**Proof.** Since \( V(t, x, p) \in \mathcal{D}_1 \), \( \bar{q}(t, x, p) > 0 \), the optimal control \( q^*(t, x, p) \) of (8) (10) is identical to \( \bar{q} \) given by (10) in the region \( \mathcal{D}_1 \). Substituting it into (8) and simplifying the expression yields (12). \( \Box \)

**Lemma 3.2.** Let \( \mathcal{D}_2 := \{(t, x, p) : \bar{q} \leq 0\} \). Suppose \( V(t, x, p) \) satisfies the properties that \( V_x(t, x, p) > 0 \) and \( V_{xx}(t, x, p) < 0 \) is a solution to

\[
V_t + (rx + \theta a)V_x = \frac{1}{2} b V_{xx} - \frac{1}{2} \left( \frac{(\mu - r)^2}{\sigma^2 p^{2\beta}} \right) \frac{V_x^2}{V_{xx}} - p(\mu - r) \frac{V_x V_{px}}{V_{xx}} \\
+ \mu p V_p + \frac{1}{2} \sigma^2 p^{2\beta+2} V_{pp} - \frac{1}{2} \sigma^2 p^{2\beta+2} \frac{V_{px}^2}{V_{xx}} = 0 \tag{14}
\]

with terminal condition

\[
V(T, x, p) = \mathcal{U}(x). \tag{15}
\]

Then \( V(t, p, x) \) satisfies the HJB equation (8) on \( \mathcal{D}_2 \) with boundary condition \( V(T, x, p) = \mathcal{U}(x) \).

**Theorem 3.1** (Verification theorem) Suppose \( \mathcal{J} \in C^{1,2,2} \) be a concave solution to the HJB equation (8) subject to the boundary condition \( V(T, x, p) = \mathcal{U}(x) \). Then the value function \( V \) is given by

\[
\mathcal{J}(t, x, p) = V(t, x, p). \tag{16}
\]

Moreover, let \( \pi^* = (q^*, \zeta^*) \in \mathcal{M}^F \) be such that

\[
\mathcal{A}^{\pi^*} V(t, x, p) = 0. \tag{17}
\]

Then the policy \( \pi^* \) is the optimal strategy.

**Proof.** Since the proof is standard, we omit the details here. \( \Box \)
3.2 Convex Legendre dual of $V$

Since $V$ is strictly concave with respect to $x$ in its continuation region, we can define its convex dual $\hat{V}$ by the Legendre transform:

$$\hat{V}(t, z, p) = \max_{x > 0} \{ V(t, p, x) - zx \}. \quad (18)$$

From (18), it follows that the critical value $x^*$ solves $z = V(t, z, p)$. Let the value of $x^*$ where this optimum is attained is denoted by $I(t, z, p)$, so that

$$I(t, z, p) = \inf_{v > 0} \{ v | V(t, x, p) \geq zv + \hat{V}(t, z, p) \}, \quad (19)$$

The function $\hat{V}$ is related to $I$ by $I = -\hat{V}_z$, so we can take either one of the two function $g$ and $\hat{V}$ as the dual of $V$. Obviously, we have

$$V_x = z \quad (20)$$

and hence

$$\hat{V}(t, z, p) = V(t, x, I) - z I, \quad I(t, z, p) = x. \quad (21)$$

**Theorem 3.2** $\hat{V}$ equals the maximum expect utility $V$ on $\{(t, x, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \}$, and the reinsurance and investment policy $\pi^* = (q^*, \zeta^*)$ given in feedback form by $\pi^* = \pi^*(t, Z^*, P)$ is an optimal policy, in which $Z^*_t$ is the optimally controlled wealth and the function $\pi^*$ is given by

$$\begin{cases} 
q^* = \left(1 - \frac{\eta}{\sigma^2 p^2} z I_z \right) I_p I(t, x, p), \\
\zeta^* = -\frac{(\mu - r) z I_z + \sigma^2 p^2 b^2 + 1} {\sigma^2 p^2 b^2 I}. 
\end{cases} \quad (22)$$

**Proof.** By differentiating (20) and (21) with respect to $t, p$ and $z$, we obtain the following expressions:

$$V_t = \hat{V}_t, \quad V_p = \hat{V}_p, \quad V_x = z, \quad \hat{V}_z = -I$$

$$V_{pp} = \hat{V}_{pp} - \frac{\hat{V}^2_{pz}}{\hat{V}_{zz}}, \quad V_{xx} = -\frac{1}{\hat{V}_{zz}}, \quad V_{px} = -\frac{\hat{V}_{pz}}{\hat{V}_{zz}}. \quad (23)$$

Substituting (20) and (21) and the above expression (23) into (11), and differentiating $V$ with respect to $z$, we derive:

$$I_t = r I - (\theta - \eta) a + r pi_p + \frac{(\mu - r)^2} {\sigma^2 p^2 b^2} z - rz) I_z + \frac{1}{2} \sigma^2 p^2 b^2 I_{pp} - (\mu - r) p z I_{pz} + \frac{1}{2} \frac{(\mu - r)^2} {\sigma^2 p^2 b^2} z^2 I_{zz} = 0. \quad (24)$$
on $D_1$, and
\begin{align}
I_t &= rI - \theta a + rpI_p + \left(\frac{(\mu - r)^2}{\sigma^2 p^{2\beta}} z - rz\right)I_z
\nonumber \\
& \quad + \frac{1}{2} \sigma^2 p^{2\beta + 2} I_{pp} - (\mu - r)p z I_{pz} + \frac{1}{2} \left(\frac{(\mu - r)^2}{\sigma^2 p^{2\beta}}\right)z^2 I_{zz} = 0,
\end{align}
onumber
(25)
onumber
	on $D_2$. Since $\hat{V}$ is strictly convex with respect to $z$, the optimal policy $\pi^*$ in (24) is given by the first-order necessary condition, which results in the expression in (22). □

### 3.3 Explicit solutions to the optimization problem

From (7) and Remark 3.2, we can derive the following boundary condition:
\begin{equation}
I(T, z, p) = \frac{1 - \gamma}{\kappa} (z^{-\frac{1}{\gamma}} - \varsigma).
\end{equation}
(26)

**Lemma 3.3.** The solution $I(t, p, z)$ of the non-linear PDE (24) with the terminal condition (26) is given by
\begin{align}
I(t, z, p) &= \frac{1 - \gamma}{\kappa} f(t, s)(z^{-\frac{1}{\gamma}} - \varsigma) + h(t, s) + n(t).
\end{align}
(27)

where
\begin{align}
n(t) &= \left[ -\frac{(\theta - \eta)a}{r} (1 - e^{-r(T-t)}) \right] I_{D_1}(t, x, p) + \left[ -\frac{\theta a}{r} (1 - e^{-r(T-t)}) \right] \times I_{D_2}(t, x, p),
\end{align}
(28)
\begin{align}
f(t, p) &= A(t)e^{B(t)p^{-2\beta}},
\end{align}
(29)
\begin{align}
h(t, p) &= H(t)p^{-2\beta} + K(t).
\end{align}
(30)

**Here**
\begin{align}
A(t) &= \left[ e^{(\rho_1(2\beta+1) + \frac{\rho_1}{1 - \gamma})} + 2\frac{\gamma}{\sqrt{\gamma}} (T-t) \right] I_{D_1}(t, x, p)
\nonumber \\
& \quad + \left[ e^{(\rho_1(2\beta+1) + \frac{\rho_1}{1 - \gamma})} (T-t) \right] \times \left( \frac{\rho_2 - \rho_1}{\rho_2 - \rho_1 e^{2\beta/(\rho_1 - \rho_2)(T-t)}} \right) \frac{2\beta + 1}{2\beta} I_{D_2}(t, x, p)
\end{align}
(31)
\begin{align}
B(t) &= \sigma^{-2} \rho_1 - \rho_1 e^{2\beta/(\rho_1 - \rho_2)(T-t)}
\nonumber \\
& \quad + \frac{\rho_1 e^{2\beta/(\rho_1 - \rho_2)(T-t)}}{1 - \rho_1 e^{2\beta/(\rho_1 - \rho_2)(T-t)}}
\end{align}
(32)

with
\begin{align}
\rho_{1,2} = \frac{(\mu - r\gamma) \pm \sqrt{(1 - \gamma)(\mu^2 - r^2\gamma)}}{2\beta(1 - \gamma)}.
\end{align}
(33)
and
\[
H(t) = \frac{e^{(2\beta+1)rt}}{\kappa} \int_t^T e^{-(2\beta+1)rs} (1 - \gamma)A(s)e^{B(s)p^{-2\beta}} \times (B'(s) - 2\beta r B(s)) \, ds,
\]
(34)
and
\[
K(t) = -e^{rt} \int_t^T \frac{1}{\kappa} e^{-rt} \left\{ (1 - \gamma)A(s)B(s) \right\} \, ds.
\]
(35)

**Proof.** See Appendix. □

The next goal is to derive an explicit expression for the optimal strategy \(\pi^*((t)) = (q^*(t), \zeta^*(t))\).

**Theorem 3.3.** The optimal investment strategy \(\pi^*(t) = (q^*(t), \zeta^*(t))\) is given by
\[
\begin{align*}
q^*(t) &= \left\{ 1 - \frac{\alpha}{\nabla^2} \left[ t + \frac{\kappa}{f(1-\gamma)} (x - tH - n) \right] f \right\} T(t, x, p), \\
\zeta^*(t) &= \frac{\alpha e^{-(x-tH-n)}}{x(1-\gamma)p^{2\beta}} + \frac{-2\beta p^2 p^{2\beta+1} t h_p}{x(1-\gamma)p^{2\beta}}.
\end{align*}
\]
(36)

**Proof.** Substituting the derivatives \((I_z \text{ and } I_p)\) in (22), we have
\[
I(t, p, z) = \frac{1 - \gamma}{\kappa} f(t, s)(z^{\gamma-1} - t) + Ih(t, s) + n(t).
\]
(37)
From Eqs (37) and (21), we get
\[
z^{\gamma-1} = t + \frac{\kappa}{f(1-\gamma)} (x - tH - n).
\]
(38)
Since \(f = Ae^{By} = Ae^{Bp^{-2\beta}}\), we have \(f_p = -2\beta p^{-2\beta-1} AB e^{Bp^{-2\beta}}\) and so
\[
\frac{f_p}{f} = -2\beta p^{-2\beta-1} B.
\]
(39)
From (37) to (39), we take
\[
I_z = -\frac{1}{\kappa} z^{\gamma-1} f = -\frac{1}{\kappa} - \frac{1}{z} \left[ t + \frac{\kappa}{f(1-\gamma)} (x - tH - n) \right] f
\]
(40)
and
\[
I_p = -\frac{1 - \gamma}{\kappa} (z^{\gamma-1} - t) f_p + Ih_p
= \frac{f_p}{f} (x - tH - n) + Ih_p = -2\beta p^{-2\beta-1} B(x - tH - n) + Ih_p.
\]
(41)
Putting (40) and (41) in (22), we obtain (36) and the proof is complete. □
A Proof of Lemma 3.3

Proof. We try to find a solution of (24) in the form (27) with the boundary conditions given by \( n(T) = 0, f(T, p) = 1 \) and \( h(T, p) = 0 \). Then, we substituting the derivatives \((I_t, I_p, I_z, I_{pz})\) into (24), and we can derive the following three equations on \( D_1 \)

\[
\begin{align*}
\frac{1-\gamma}{\kappa} f_t + \frac{1-\gamma}{\kappa} r_p f_p - \frac{1}{2\kappa} \sigma^2 p^2 t^{\beta+2} f_{pp} - \frac{1}{\kappa} r f - \frac{f}{\kappa} \left( \frac{\mu - r}{\sigma^2 p^2} + \frac{a^2 \eta^2}{b^2} - r \right) \\
- \frac{f}{\kappa} \left( \frac{2\gamma}{\kappa} - 1 \right) \left( \frac{\mu - r}{2\sigma^2 p^2} + \frac{a^2 \eta^2}{2b^2} \right) + \frac{\mu - r}{\kappa} p f_p = 0. \\
- \frac{1-\gamma}{\kappa} f_t + h_t - \frac{1-\gamma}{\kappa} r p f_p + r p h_p \\
- \frac{1-\gamma}{2\kappa} \sigma^2 p^2 t^{\beta+2} f_{pp} \frac{1}{2} \sigma^2 p^2 t^{\beta+2} h_{pp} + \frac{1-\gamma}{\kappa} r f - r h = 0
\end{align*}
\]

and

\[
n(t)' - rn(t) - (\theta - \zeta)a = 0.
\]

First, we solve the ODE (42). To do this, put

\[
f(t, p) = \phi(t, y) \text{ and } y = p^{-\beta}
\]

with the boundary condition given by \( \phi(T, y) = 1 \). Then

\[
f_t = \phi_t, \quad f_s = -2\beta y p^{-1} \phi_y, \quad f_{pp} = 4\beta^2 y^2 p^{-2} \phi_{yy} + (4\beta^2 + 2\beta) y p^{-2} \phi_y.
\]

Substituting these derivatives into (42), we have

\[
\begin{align*}
\frac{1-\gamma}{\kappa} \phi_t + \frac{1-\gamma}{\kappa} r p (-2\beta) y p^{-1} \phi_y + \frac{1-\gamma}{2\kappa} \sigma^2 y^{-1} p^2 [4\beta^2 y^2 p^{-2} \phi_{yy} + (4\beta^2 + 2\beta) y p^{-2} \phi_y] \\
- \frac{1-\gamma}{\kappa} r \phi - \frac{\phi}{\kappa} \left( \frac{\mu - r}{\sigma^2} y + \frac{a^2 \eta^2}{b^2} - r \right) - \frac{f}{\kappa} \left( \frac{2\gamma}{\kappa} - 1 \right) \left( \frac{\mu - r}{2\sigma^2} y + \frac{a^2 \eta^2}{2b^2} \right) \\
+ \frac{\mu - r}{\kappa} (-2\beta) y p^{-1} \phi_y = 0.
\end{align*}
\]

We guess that a solution of (46) in the form \( \phi(t, y) = A(t) e^{B(t)y} \) with \( A(T) = 1 \) and \( B(T) = 0 \). Substituting this function in (46) and multiplying \( \frac{\kappa}{(1-\gamma)A} e^{-B y} \), we obtain

\[
\begin{align*}
\frac{A'}{A} + \beta(2\beta + 1) \sigma^2 B + \frac{r \gamma}{1-\gamma} + \frac{a^2 \eta^2}{2b^2} \frac{\gamma}{(1-\gamma)^2} \\
+ y \left[ B' + \frac{2\beta(r \gamma - \mu)}{1-\gamma} B + 2\sigma^2 \beta^2 B^2 + \frac{(\mu - r)^2 \gamma}{2\sigma^2 (1-\gamma)^2} \right] = 0
\end{align*}
\]

We can split this equation into two ODEs as follows:

\[
\begin{align*}
\frac{A'}{A} + \beta(2\beta + 1) \sigma^2 B + \frac{r \gamma}{1-\gamma} + \frac{a^2 \eta^2}{2b^2} \frac{\gamma}{(1-\gamma)^2} = 0, \quad A(T) = 1, \quad (48)
\end{align*}
\]
Taking into account the boundary conditions, the solutions to Eqs. (48) and (49) are given by (31) and (32).

Taking into account the boundary condition \( n(T) = 0 \), we can easily check that the solution of the Eq. (44) is given by

\[
n(t) = -\frac{(\theta - \eta)a}{r} (1 - e^{-r(T-t)}).
\]

Finally, we solve Eq. (46). Let

\[
h(t, p) = H(t)y + K(t) \quad \text{and} \quad y = p^{-2\beta}
\]

with the boundary condition given by \( H(T) = 0 \) and \( K(T) = 0 \). Then

\[
h_t = H'y + K', \quad h_p = -2\beta p^{-1}yH, \quad h_{pp} = (4\beta^2 + 2\beta)p^{-2}yH.
\]

Substituting these derivatives and \( \phi = Ae^{By} \) into PDE (46), we have

\[
-\frac{1-\gamma}{\kappa} A'e^{By} + K' - \frac{1-\gamma}{2\kappa} \sigma^2 (4\beta^2 + 2\beta)ABe^{By} + (2\beta^2 + \beta)\sigma^2 H + \frac{1-\gamma}{\kappa} rAe^{By}
\]

\[
- rH + y \left[ -\frac{1-\gamma}{\kappa} A'e^{By} + H' + \frac{1-\gamma}{\kappa} 2\beta rABe^{By} - \frac{1-\gamma}{\kappa} 2\sigma^2 \beta^2 AB^2 e^{By} \right.
\]

\[
\left. -(2\beta + 1)rH \right] = 0.
\]

We can split this equation into two ordinary differential equations as follows:

\[
-\frac{1-\gamma}{\kappa} AB'e^{By} + H' + \frac{1-\gamma}{\kappa} 2\beta rABe^{By} - \frac{1-\gamma}{\kappa} 2\sigma^2 \beta^2 AB^2 e^{By}
\]

\[
\left. -(2\beta + 1)rH \right] = 0, \quad H(T) = 0.
\]

\[
-\frac{1-\gamma}{\kappa} A'e^{By} + K' - \frac{1-\gamma}{2\kappa} \sigma^2 (4\beta^2 + 2\beta)ABe^{By} + (2\beta^2 + \beta)\sigma^2 H
\]

\[
+ \frac{1-\gamma}{\kappa} rAe^{By} - rH = 0, \quad K(T) = 0.
\]

It is easy to see that the solutions for (53) and (54) are given by (34) and (35). Similar approach can applied on the set \( \mathcal{D}_2 \) leading to the result in Lemma 3.3. Then the proof of Lemma 3.3 is complete. \( \square \)
References


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