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Measuring Fuzzy Specificity Using Fuzzy Unit Hypercube

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Abstract

In this article, we propose a new measure of specificity using Kosko's fuzzy hypercube. We show that specificity of a fuzzy set can be measured as the distance between the set and a point on the principal diagonal of the hypercube. It is established that, Yager's Linear measure of specificity is a particular case of this measure. Some interesting geometrical view of ambiguity measures in Yager-Fishburn framework is established as well.

Keywords: Specificity, Fuzzy hypercube, Principal diagonal, Linear measure

1 Introduction

Two types of uncertainty in fuzzy sets can easily be identified: fuzziness, which is related to lack of clear cut boundaries and specificity, related to precision of a fuzzy set [9]. Specificity and fuzziness refer to two distinct characteristics of fuzzy sets. Specificity measures the degree of having just one element in a fuzzy set [9], whereas fuzziness measures the extent to which a fuzzy set is not crisp [3]. To measure lack of specificity, Krill and Higashi [2] have proposed the concept of non-specificity.

The concept of specificity for fuzzy sets was first introduced by Yager in 1982 [4] as tool for measuring tranquility in decision making process. The specificity-correctness trade-off principle, which plays an important role in information theory has been introduced in [9]. Specificity is useful in expert systems and knowledge based systems. In such systems, it has been shown that the output information should be both specific and correct to be useful [9]. In [6], Yager has suggested the use of specificity in default reasoning .

Specificity can also be used to indicate the degree to which a possibility distribution allows one and only one element as its manifestation [7]. In this context, it provides a measure of the amount of information contained in a possibility distribution [7]. Thus, specificity plays a role in possibility theory similar to that of Shannon entropy in probability theory. As such, measures of specificity have been largely formulated and investigated within the framework of possibility theory [7, 1, 5]. Extending this investigation to other frameworks allows derivation of different expression of measures of specificity for application in different settings.

The purpose of this paper is to extend Yager's geometrical view of specificity[9]. In particular, we show that specificity of a fuzzy set can be measured as the distance between the set and a point on the principal diagonal of a fuzzy hypercube. We shall see that this measure provides a geometrical interpretation of Yager's linear measure of specificity. Further, it is demonstrated that particular cases of this measure gives a geometrical perspective of measures of ambiguity suggested in [9]. The subsequent sections of this paper are organized as follows: Section 2 sheds light on the theory of fuzzy sets including their geometrical representation. Section 3 reviews measures of specificity. A new measure of specificity is given in section 4.

2 Preliminaries

A fuzzy set A over a finite universe $X = \{x_1, x_2, \ldots, x_n\}$ is characterized a mapping

$$\mu_A : X \to [0,\ 1] \tag{1}$$

such that

$$\mu_A(x_i) = a_i \in [0, 1] \tag{2}$$

called its membership function. Sets of all fuzzy and crisp sets will be denoted by $\mathcal{F}(X)$ and $\mathcal{P}(X)$, respectively. Crisp sets

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\} \tag{3}$$

and

$$A_{\underline{\alpha}} = \{x \in X | \mu_A(x) > \alpha\} \tag{4}$$

are called alpha and strong alpha cuts, respectively. Fuzzy set A is said to be normal if

$$\max(\mu_A(x)) = 1 \tag{5}$$

A fuzzy set A such that $\mu_A(x_i) = a \forall i$ will be denoted by $[a]$. The complement A^c of A is specified by the membership function

$$\mu_{A^c}(x) = 1 - \mu_A(x) \tag{6}$$

Fuzzy operators of the union, $A \cup B$ and intersection, $A \cap B$ of two fuzzy A and B are specified by the membership functions

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \tag{7}$$

and

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \tag{8}$$

respectively. The cardinality or size of fuzzy set A, denoted $|A|$ is the sum of its membership values, that is,

$$|A| = \sum_{i=1}^{n} \mu_A(x_i) \tag{9}$$

A fuzzy set A is said to be a subset of set B if and only if $\mu_A(x) \leq \mu_B(x) \forall x$. Fuzzy set

$$A = ((x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \ldots, (x_n, \mu_A(x_n))) \tag{10}$$

can be geometrically represented by a fit vector $(a_1, a_2, \ldots, a_n)$ in an n-dimensional unit cube

$$[0, 1] \times [0, 1] \times \cdots \times [0, 1] = [0, 1]^n = I^n, \quad n \geq 1 \tag{11}$$

The cube I^n is commonly known as the fuzzy unit hypercube [3]. The fit value a_i corresponds to the membership grade in the i^{th} dimension. Vertices of the hypercube correspond to the 2^n non-fuzzy sets. Fuzzy sets reside inside and on the surface of the hypercube. For an n-dimensional fuzzy hypercube, the principal diagonal is the diagonal joining the origin $\emptyset = (0, 0, \ldots, 0)$ and the vertex corresponding to whole space $X = \{1, 1, 1, \ldots, 1\}$. Observe that any point on the principal diagonal is a constant fuzzy set, $[a]$.

3 Fuzzy Specificity

Consider a proposition of the form " P is A" where A is a fuzzy subset of the domain X. Zadeh [7] showed that such proposition induces a possibility on X such that for any element x in X, its membership value $\mu_A(x)$ is a possibility that x is the value of P. In this case, A is said to be a possibility distribution associated with variable P.

Yager [7] viewed specificity of a possibility distribution as the degree to which the distribution has one element as its possible manifestation. It measures the degree of the truth in the proposition "A is a one element set". Formally, a measure of specificity $Sp(A)$ of fuzzy set A is a mapping $Sp : \mathcal{F}(X) \to [0, 1]$ [9] satisfying the following axioms: $\forall A \in \mathcal{F}(X)$

$(S1)$ $Sp(\emptyset) = 0$

$(S2)$ $Sp(A) = 1$ iff A is a singleton non-fuzzy set.

$(S3)$ $Sp(A)$ is strictly increasing with respect to a_j and strictly decreasing with respect to $a_j \forall j \geq 2$.

where a_j is the j^{th} largest membership grade in A. The first and second requirements provide boundary conditions for specificity. The last property indicates that specificity increases as the largest membership grade increases and decreases as any of the non maximal membership grades increases.

A measure of specificity Sp is said to be regular if $Sp(A) = 0$ for any constant fuzzy set $A = [a]$ [9]. In other words, we are least certain when we have to make a decision, but none of the decision choices is a better alternative. If Sp and $Sp*$ are measures of specificity then Sp^* is said to be stricter than Sp if $Sp^*(A) \geq Sp(A)$ for all fuzzy subsets A of X [9].

In [4] Yager has proposed the following formula for measuring specificity

$$Sp(A) = \int_0^{h(A)} \frac{1}{|A_\alpha|} d\alpha \tag{12}$$

where $h(A) = \mathbf{Max}\{\mu_A(x)\}$ is the height of fuzzy set A. It can be easily checked that this expression satisfies the requirements stipulated in the above definition. Dubois and Prade [1] have given a probabilistic interpretation of this measure.

Yager [9] has introduced the product measure of specificity based on the perspective of a multi-criterion decision making problem. In this formulation, specificity of fuzzy set A is defined as

$$Sp(A) = a_1 \prod_{j=2}^{n} (ra_j + (1 - a_j)) \tag{13}$$

This measure determines the extent to which A is a singleton set.

Specificity can also be intuitively measured as the inverse of the distance between a fuzzy set and the nearest singleton set[9]. That is, if we let $B_k \in \mathcal{P}(X)$ be a singleton set with 1 in the k^{th} position, $1 \leq k \leq n$, then specificity of A is given as

$$Sp(A) = 1 - \mathbf{Min}\lambda_p(A, B_k) = 1 - \lambda_p(A, B_j), j = k \tag{14}$$

where

$$\lambda_p(A, B_k) = \begin{cases} l_p(A, B_k), & \text{if} \quad l_p(A, B_k) \leq 1 \\ 1, & \text{if} \quad l_p(A, B_k) > 1 \end{cases} \tag{15}$$

and $l_p(A, B)$ is the Minkowski class of metrics

$$l^p(A, B) = \left[\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \right]^{\frac{1}{p}}, \quad 1 \leq p \leq \infty \tag{16}$$

Another important class of measures is the so called linear measure of specificity [9].

$$Sp(A) = a_1 - \sum_{j=2}^{n} w_j a_j \tag{17}$$

where w_j is a set of weights satisfying the following properties:

(i) $w_j \in [0, 1]$

(ii) $\sum_{j=2}^{n} w_j = 1$

(iii) $w_j \geq w_i \forall 1 < j < i$

If $w_j = \frac{1}{n-1} \quad forall \quad j \geq 2$, then we have the less stricter linear measure which is simply the difference between the largest membership grade a_1 and the average of the rest of the membership values of A.

$$Sp(A) = a_1 - \frac{1}{n-1} \sum_{j=2}^{n} a_j \tag{18}$$

In addition, this class of measures is regular with $Sp\,(A) = a_1 - a_2$ as the the strictest case.

4 Measuring specificity using a fuzzy hypercube

In this section we extend Yager's geometrical view of measures of that makes use of a fuzzy unit hypercube [9]. In particular, we provide a geometrical interpretation of Yager's linear measure of specificity.

Suppose $w_j = \frac{1}{n-1} \forall j$ and let the membership grades of fuzzy set A be decreasingly ordered, $a_1 \geq a_2 \geq \cdots \geq a_n$. Yager's linear measure can be checked to be equal to

$$Sp\,(A) = \frac{1}{n-1}\left(|a_1 - a_1| + |a_1 - a_2| + |a_1 - a_3| + \cdots + |a_1 - a_n|\right) \tag{19}$$

which is the fuzzy Hamming distance between set A and the point $[a_1]$ on the principal diagonal,

$$Sp\,(A) = \frac{1}{n-1} l_1\left([a_1], A\right) \tag{20}$$

We can now investigate a measure of specificity expressed as

$$Sp\,(A) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\left([a_1], A\right) \tag{21}$$

This formula is defines a measure of specificity if it satisfies the basic properties of specificity. The first two conditions are readily satisfied.

$$Sp\,(A) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\left([0], \emptyset\right) = 0 \tag{22}$$

and for a singleton set A

$$Sp\,(A) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\left([1], A\right) = 1 \tag{23}$$

If the largest membership grade a_1 increases then $d_p\left([a_1], A\right)$ increases. On the other hand, increasing non maximal membership grade decreases this metric. This completes the prove. Further, observe that this measure is regular since

$$Sp\,(A) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\left([a_1], A\right) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\left([a], A\right) = 0 \tag{24}$$

for any constant fuzzy set $A = [a]$. A more general form of this measure is given as

$$Sp\,(A) = \left(\sum_{j=2}^{n} |a_1 T - w_j a_j|^p \right)^{\frac{1}{p}} \tag{25}$$

where $T = \frac{1}{n-1}$. We obtain more interesting results by considering the following particular cases
For $p = 1$:

$$\begin{aligned} Sp\,(A) &= |a_1 T - w_2 a_2| + |a_1 T - w_3 a_3| + \cdots + |a_1 T - w_n a_n| \\ &= a_1\,(n-1)\,T - \sum_{j=2}^{n} w_j a_j = a_1 - \sum_{j=2}^{n} w_j a_j \end{aligned} \tag{26}$$

and when $p = \infty$

$$Sp\,(A) = l^{\infty}\,([a_1]\,, A) = \mathbf{Max}\,\{|a_1 - w_2 a_2|, |a_1 - w_3 a_3|, \ldots, |a_1 - w_n a_n\} = a_1 - w_n a_n \tag{27}$$

It can be easily checked that the expression in (26) is not a measure of specificity since if we take $A = (1, 1, 1, 0)$ then

$$Sp\,(A) = l_{\infty}\,([a_1]\,, A) = a_1 - w_n a_n = 1 - 0 = 1 \tag{28}$$

However, Yager [8] has shown that (26) defines a measure of ambiguity in the framework of Fishburn which is expressed as

$$\beta\,(A) = Sp\,(A) + Sp\,(A^c) = a_1 - a_n \tag{29}$$

where the measure of ambiguity β is a mapping $\beta : \mathcal{F}\,(X) \to [0, 1]$ varifying the following properties[8]:

(i) $\beta\,(\emptyset) = 0$

(ii) $\beta\,(A) = \beta\,(A^c)$

(iii) $\beta\,(A \cap B) + \beta\,(A \cup B) \leq \beta\,(A) + \beta\,(B)$

From the definition of A^c, we can also see that

$$\begin{aligned} &|\,(1 - a_n) - (1 - a_n)\,| + |\,(1 - a_n) - (1 - a_{n-1})\,| + |\,(1 - a_n) - (1 - a_{n-2})\,| + \cdots + \\ &\qquad |\,(1 - a_n) - (1 - a_1)\,| = |a_1 - a_n| + |a_2 - a_n| + \cdots + |a_{n-1} - a_n| \end{aligned} \tag{30}$$

and thus

$$Sp\,(A^c) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\,([1 - a_n]\,, A^c) = \frac{1}{(n-1)^{\frac{1}{p}}} l_p\,([a_n]\,, A) \tag{31}$$

For $p = 1$ we have

$$Sp\left(A\right) = \frac{1}{n-1}\left(\sum_{j=1}^{n-1} a_j\right) - a_n \tag{32}$$

Similarly for $p = \infty$ we have

$$Sp\left(A^c\right) = a_1 - a_n \tag{33}$$

Thus for $p = \infty$, $Sp\left(A\right) = Sp\left(A^c\right)$. This equality is in agreement with the characterization of measures of ambiguity in the framework of Fishburn [8]. This measure can also be intuitively defined as

$$\beta\left(A\right) = \frac{1}{n^{\frac{1}{p}}} l_p\left(\left[a_1\right], \left[a_n\right]\right), 1 \leq p \leq \infty \tag{34}$$

From this analysis we conclude that the expression given in (25) is a measure of specificity if P is finite.

5 Conclusion

We have shown that specificity of a fuzzy set can be measured as the distance between the set and a point on the principal diagonal of a fuzzy hypercube. This view gives an interesting geometrical interpretation of Yager's linear measure of specificity. In addition, we have demonstrated that particular cases of this measure gives a geometrical perspective of measures ambiguity in the context of Yager-Fishburn

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