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Applying Pade Approximation Model in Optimal Control Problem for a Distributed Parameter System with Time Delay

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Abstract

This paper presents a research replacing a delayed object by using first-order Pade approximation model in order to solve the problem of optimal control for a distributed parameter system with time delay (DPSTD). The system is applied to a specific one-sided heat-conduction system in a heating furnace to control temperature for a slab following the most accurate burning standards.
The aim of problem is to find an optimal control signal so that the error between the distribution of real temperature of the object and the desired temperature is minimum after a given period of time *T*.

Keywords: optimal control, distributed parameter systems, delay, numerical method, Pade approximation

1. Introduction

Theoretically, Pade approximation has been studied for a long time and mainly applied in finding solutions of differential algebraic equations.
Pade approximaton can offer a function approximation having more advantages than Taylor expansion, especially with the objects having large time delay compared to its time constant. The paper presents a research replacing a delayed object by using first-order Pade approximation model in order to solve the problem of optimal control for a distributed parameter system with time delay (DPSTD), typically a controlled object described by heat transfer equation, which is one of the physical processes with distributed parameters.

2. Pade Approximation Method

Suppose the function *f(x)* is expanded under an exponential sequence:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots \quad (1)$$

Move the function into Pade' expansion as follows:

$$a_0 + a_1 x + a_2 x^2 + \dots = \frac{P_m(x)}{Q_n(x)} = \frac{p_m x^m + p_{m-1} x^{m-1} + \dots + p_1 x + p_0}{q_n x^n + p_{n-1} x^{n-1} + \dots + q_1 x + 1} \quad (2)$$

The Pade approximation must be satisfied so that if the analysis of the right-hand side (2) according to the Taylor sequence at near zero-point, its first element ($m+n$) is coincided with ($m+n$) elements of the left-hand side: a_0, a_1, ……[1]

In order to get highly accurate approximation, $m = n$ or $m=n+1$ is ussually chosen [2]. After analysing as (2), we need determine ($m+n+1$) unknowns: $p_0, p_1,\dots,p_m$; $q_0, q_1,\dots,q_n$.. From Eq. (2), we offer a system of equations to find these unknowns as follows:

$$\begin{cases} a_0 - p_0 = 0 \\ q_1 a_0 + a_1 - p_1 = 0 \\ q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0 \\ q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 = 0 \\ \dots\dots \\ q_n a_{m-n} + q_{n-1} a_{m-n+1} + \dots + a_m - q_n = 0 \end{cases} \quad (3)$$

$$\begin{cases} q_n a_{m-n+1} + q_{n-1} a_{m-n+2} + \dots + q_1 a_m + a_{m+1} = 0 \\ q_n a_{m-n+2} + q_{n-1} a_{m-n+3} + \dots + q_1 a_{m+1} + a_{m+2} = 0 \\ \dots\dots \\ q_n a_m + q_{m-1} a_{n+1} + \dots + q_1 a_{m+n-1} + a_{m+n} = 0 \end{cases} \quad (4)$$

After solving the system (4), we get: $q_0, q_1,\dots,q_n$, , then replace into sysytem (3), we can find: $p_0, p_1,\dots,p_m$

An delayed object in the forms of $e^{-\rho s}$ can be expanded into an exponential sequence:

$$e^{-\rho s} = \sum_{k=0}^{\infty} \frac{(-\rho s)^k}{k!} \tag{5}$$

Thus, compare with the left-hand side of (2), we have:

$$a_0 = 1; a_1 = -\rho; a_2 = \frac{\rho^2}{2}; ... a_k = \frac{(-\rho)^k}{k!} \tag{6}$$

Substituting the coefficients a_k (*k=0,1,2...*) from (5) and (6) into (3) and (4). Solving these equation systems, we obtain the result of Pade approximation, in the case, *m* = *n* as follows:

$$e^{-\rho s} = \frac{\sum_{k=0}^{n} H(k)(-\rho s)^k}{\sum_{k=0}^{n} H(k)(\rho s)^k} \tag{7}$$

where
$$H(k+1) = \frac{n-k}{(2n-k)(k+1)} H(k) \tag{8}$$

with: *H(0)* = 1; *n*: a number of orders need to be replaced
- With *n* = 1, we have the first –order Pade approximation:

$$e^{-\rho s} \approx \frac{1 - \frac{\rho s}{2}}{1 + \frac{\rho s}{2}} \tag{9}$$

- With *n* = 2, we have the second–order Pade approximation:

$$e^{-\rho s} \approx \frac{1 - \frac{\rho s}{2} + \frac{\rho^2 s^2}{12}}{1 + \frac{\rho s}{2} + \frac{\rho^2 s^2}{12}} \tag{10}$$

-With *n* =3,4,…,we have higher-order Pade approximations
Consider an object with time delay:

$$\gamma \frac{du(t)}{dt} + u(t) = k\mathrm{w}(t - \rho) \tag{11}$$

where γ is the time constant and ρ is the time delay of the object, *k* is gain coefficient. We have conducted simulation by Matlab Simulink for these approximate forms. In the process of simulation, we changed the ratio γ/ρ by keeping γ = constant and ρ changed, finally we draw the following conclusions:

- If $\gamma/\rho \geq 10$, we should use expansion following exponential sequence
- If $6 \leq \gamma/\rho < 10$, we should use Pade expansion with $n = 1$
- If $2 \leq \gamma/\rho < 6$, we should use Pade expansion with $n = 2$

3. The problem of optimal control

3.1. The object model

As a typical distributed parameter system, the one-side heat conduction system in a furnace is considered. The process of one-side heating of the object in a furnace is described by the partial differential equation, as follows [3],[4], [6]:

$$a\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t} \tag{12}$$

where $q(x,t)$, the temperature distribution in the object, is the output needing to be controlled, depending on the spatial coordinate x with $0 \leq x \leq \delta$ and the time t with $0 \leq t \leq T$, a is the temperature-conducting factor (m^2/s), δ is the thickness of object (m), T is the allowed burning time (s)

The initial and boundary conditions are given in [3],[4],[6].

$$q(x,0) = 0 \tag{13}$$

$$\lambda \left.\frac{\partial q(x,t)}{\partial x}\right|_{x=0} = \alpha\left[q(0,t) - u(t)\right] \tag{14}$$

$$\lambda \left.\frac{\partial q(x,t)}{\partial x}\right|_{x=\delta} = 0 \tag{15}$$

with α as the heat-transfer coefficient between the furnace space and the object ($w/m^2\,.^0C$), λ as the heat-conducting coefficient of material ($w/m\,.^0C$), and $u(t)$ as the temperature of the furnace respectively (^0C).

The relationship between the provided power for the furnace $w(t)$ and the temperature of the furnace $u(t)$ is ussually the first order inertia system with time delay [4].

Hence, the relationship between $w(t)$ and $u(t)$ is described as the following equation:

$$\gamma\frac{du(t)}{dt} + u(t) = k\mathrm{w}(t-\rho) \tag{16}$$

where γ is the time constant, ρ is the time delay; k is the static transfer coefficient; $u(t)$ is the temperature of the furnace and $w(t)$ is the provided power for the furnace (controlled function of the system).

The temperature *u(t)* of the furnace is controlled by power *w(t)*, the temperature distribution *q(x,t)* in the object is controlled by means of the fuel flow *u(t)*, this temperature is controlled by power *w(t)*. Therefore, the temperature distribution *q(x,t)* will depend on power *w(t)*.

3.2. The objective function and the constrained conditions

In this case, the problem is set out as follows: we have to determine a control function *w(t)* with ($0 \leq t \leq T$) so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object *q(x,T)* at time $t = T$. It means at the end of the heating process to ensure temperature uniformity throughout the whole material:

$$I[w(t)] = \int_0^{\delta} \left[q^*(x) - q(x,T)\right]^2 dx \rightarrow \min \quad (17)$$

The constrained conditions of the control function is:

$$A_1 \leq \mathrm{w(t)} \leq A_2 \quad (18)$$

4. The solution of problem

The process of finding the optimal solution includes 2 steps:
- S*tep 1*: Find the relationship between *q(x,t)* and the control signal *w(t)*. Namely, we have to solve the equation of heat transfer (relationship between *u(t)* and *q(x,t)*) with boundary condition type-3 combined with ordinary differential equation with time delay (relationship between *w(t)* and *u(t)*)
- S*tep 2*: Find the optimal control signal $w^*(t)$ by substituting *q(x,t)* found in the first step into the function (17), after that finding optimal solution $w^*(t)$

4.1. Find the relationship between *q(x,t)* and the control signal *w(t)*

To solve the partial differential equation (12) with the initial and the boundary conditions (13), (14), and (15), we apply the Laplace transformation method with the time parameter *t*. On applying the transform with respect to *t*, the partial differential equation is reduced to an ordinary differential equation of variable *x*. The general solution of the ordinary differential equation is fitted to the boundary conditions, and the final solution is obtained by the application of the inverse transformation.
Transforming Laplace (12), we obtained:

$$a\frac{\partial^2 Q(x,s)}{\partial x^2} = sQ(x,s) \quad (19)$$

where: $Q(x,s) = L\{q(x,t)\}$

After transforming the initial and boundary conditions (13), (14) and (15), we have:

$$\lambda \frac{\partial Q(x,s)}{\partial x}\bigg|_{x=0} = \alpha\left[Q(0,s) - U(s)\right] \tag{20}$$

$$\frac{\partial Q(x,s)}{\partial x}\bigg|_{x=\delta} = 0 \tag{21}$$

From Eq. (16), assuming the delayed object satisfy the condition: 6 ≤ γ/ρ <10. To solve this problem, the authors [4] aproximated the delayed object by the first order inertia system following Taylor approximation. This paper offers an approximation method with higher accuracy. Particularly, the first order inertia system with time delay is replaced by first-order Pade appximation, Transforming Laplace (16), we obtained:

$$(\gamma s+1)U(s) = k\mathrm{W}(s).e^{-\rho s} \approx k.\mathrm{W}(s).\frac{1-\frac{\rho}{2}s}{1+\frac{\rho}{2}s} \tag{22}$$

where

$$U(s) = L\{u(t)\}\,; \quad \mathrm{W}(s) = L\{\mathrm{w}(t)\}$$

After transforming, we have the function:

$$Q(x,s) = \frac{\mathrm{W}(s).k.\left(1-\frac{\rho s}{2}\right).ch\left((\delta-x)\sqrt{\frac{s}{a}}\right)}{(\gamma s+1).\left(1+\frac{\rho}{2}s\right).\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}.sh\left(\sqrt{\frac{s}{a}}.\delta\right)+ch\left(\sqrt{\frac{s}{a}}.\delta\right)\right)} \tag{23}$$

Putting

$$G(x,s) = \frac{k.\left(1-\frac{\rho s}{2}\right).ch\left((\delta-x)\sqrt{\frac{s}{a}}\right)}{(\gamma s+1).\left(1+\frac{\rho}{2}s\right).\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}.sh\left(\sqrt{\frac{s}{a}}.\delta\right)+ch\left(\sqrt{\frac{s}{a}}.\delta\right)\right)} \tag{24}$$

We have: *Q(x,s) = G(x,s) .W(s)* (25)

From (23), (24), according to the convolution theorem, the inverse transformation of (25) is given by

$$q(x,t) = g(x,t) * w(t) \tag{26}$$

We can write:

$$q(x,t) = \int_0^t g(x,\tau)\mathrm{w}(t-\tau)d\tau \tag{27}$$

or
$$q(x,t) = \int_0^t g(x,t-\tau)\mathrm{w}(\tau)d\tau \tag{28}$$

where
$$g(x,t) = L^{-1}\{G(x,s)\} \tag{29}$$

Therefore, if we know the function *g(x,t)*, we will be able to calculate the temperature distribution *q(x,t)* from control function *w(t)*. To find *q(x,t)* in (28), we need to find the function (29). Using the inverse Laplace transformation of function *G(x,s)* we have the following result:

$$g(x,t) = \frac{k.k_0^2\left(2+\rho k_0^2\right).Cos\left(k_0.\frac{\delta-x}{\sqrt{a}}\right).}{(2-\rho k_0^2).\left[Cos\left(k_0.\frac{\delta}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}Sin\left(k_0\frac{\delta}{\sqrt{a}}\right)\right]}.e^{-k_0^2 t} + \frac{2.k.k_1^2.Cos\left(k_1\frac{\delta-x}{\sqrt{a}}\right)}{(1-\gamma k_1^2).\left[Cos\left(k_1\frac{\delta}{\sqrt{a}}\right)-\frac{\lambda.k_1}{\alpha\sqrt{a}}Sin\left(k_1\frac{\delta}{\sqrt{a}}\right)\right]}.e^{-k_1^2 t} +$$
$$+\sum_{i=2}^{\infty}\frac{k.\alpha.\left(2+\rho.\beta_i^2\right).Cos\left(\beta_i\frac{\delta-x}{\sqrt{a}}\right)}{\lambda(1-\gamma\beta_i^2)(1-\frac{\rho}{2}\beta_i^2)\left[\frac{(\lambda+\delta\alpha)}{\lambda.\beta_i\sqrt{a}}.Sin\left(\frac{\beta_i.\delta}{\sqrt{a}}\right)+\frac{\delta}{a}.Cos\left(\frac{\beta_i\delta}{\sqrt{a}}\right)\right]}.e^{-\beta_i^2 t} \tag{30}$$

where β_i is calculated from the formula: $\beta_i = \frac{\xi_i\sqrt{a}}{\delta}$ (31)

- ξ_i is the solution of the equation:

$$\xi_i.tg\,\xi_i = \frac{\alpha.\delta}{\lambda} = B_i \tag{32}$$

- Bi is the coefficient BIO of the material

$$k_0 = \frac{1}{\sqrt{\gamma}}\ ; \quad k_1 = \sqrt{\frac{2}{\rho}} \tag{33}$$

In Eq. (30):

- α is the heat-transfer factor (w/m^2 .^{0}C)
- λ is the heat-conducting factor of object (w/m. ^{0}C)
- δ is the thickness of object (m),
- a is the temperature-conducting factor (m^2/s)
- ρ is the time delay of the furnace (s)
- γ is the time constant of the furnace (s)
- k is the static transfer coefficient of the furnace

4.2. Find the optimal control signal $w^*(t)$ by using numerical method

To find the $w^*(t)$, we have to minimize the objective function (17), it means:

$$I[w(t)] = \int_0^{\delta} [q^*(x) - q(x,T)]^2 dx \rightarrow \min \qquad (34)$$

or

$$I[w(t)] = \int_0^{\delta} \left[q^*(x) - \int_0^{T} g(x, T-\tau) \mathrm{w}(\tau) d\tau \right]^2 dx \rightarrow \min \qquad (35)$$

where $q(x,T) = \int_0^{T} g(x, T-\tau) \mathrm{w}(\tau) d\tau \qquad (36)$

and $q^*(x)$ is the desired temperature distribution; $q(x,T)$ is the real temperature distribution of the object at time $t = T$.
As calculated in [3], [4] the integral numerial method is used by applying Simson formula to the right-hand side of the objective function (34). The δ, the thickness of the object, is divided into n equal lengths (n is an even number).
Similarly, it is applied to the right-hand side of the equation (35). The period of time T is devided into m equal intervals that m is an even number, too.
Hence, the optimal control problem is here to find w^*_j in order to minimize the objective function:

$$\bar{I}[w] \cong F(w) = \delta \sum_{i=0}^{n} c_i \left(q^*_i - \sum_{j=0}^{m} a_{ij} w_j \right)^2 \qquad (37)$$

The constrained conditions of the control function are described as follows:

$$A_1 \leq \mathrm{w}_j \leq A_2 \qquad (j = 0,1,\ldots,m) \qquad (38)$$

The performance index (37) is a quadratic function of the variables w_j with constraints (38) are linear. This problem can be obtained by using numerical method after a finite number of iterations of computation.

5. The simulation results

After building the algorithms and establishing the control programs, we have proceeded to run the simulation programs to test calculating programs.

5.1. The simulation for a slab of steel

- The physical parameters of the object
  - *- The heat transfer coefficient α = 335 (w/m^2. ^{0}C)*
  - *- The heat conducting coefficient λ = 55.8 (w/m. ^{0}C)*
  - *- The temperature conducting factor a= 1.03*10^{-5} (m^2/s)*
  - *- The thickness of the object δ = 0.2 (m)*

- The parameters of the furnace
    - *The time constant* $\gamma = 1300$ *(s)*
    - *The time delay of the furnace* $\rho = 140$ *(s)*
    - *The static transfer coefficient of the furnace* $k = 6$
- The desired temperature distribution $q^{*}(x) = 1000\ {}^{0}C$
- The period of heating time $T = 7200$ (s)
- Limit the temperature of furnace $u(t) \leq 2500^{0}$C
- Limit the temperature of slab surface $q(0,t) \leq 1200\ {}^{0}$C

With these parameters, the coefficient *Bi* is calculated as follows:

$$Bi = \frac{\alpha.\delta}{\lambda} = \frac{335.0,2}{55.8} \approx 1,2 \tag{39}$$

Thus, the slab of steel is a thick object because the coefficient *Bi* is greater than 0.5. Having $6 \leq \gamma/\rho < 10$

To calculate the optimal heating process, we choose $n = 6$ and $m = 36$. After the simulation, we have result like in figure 1.

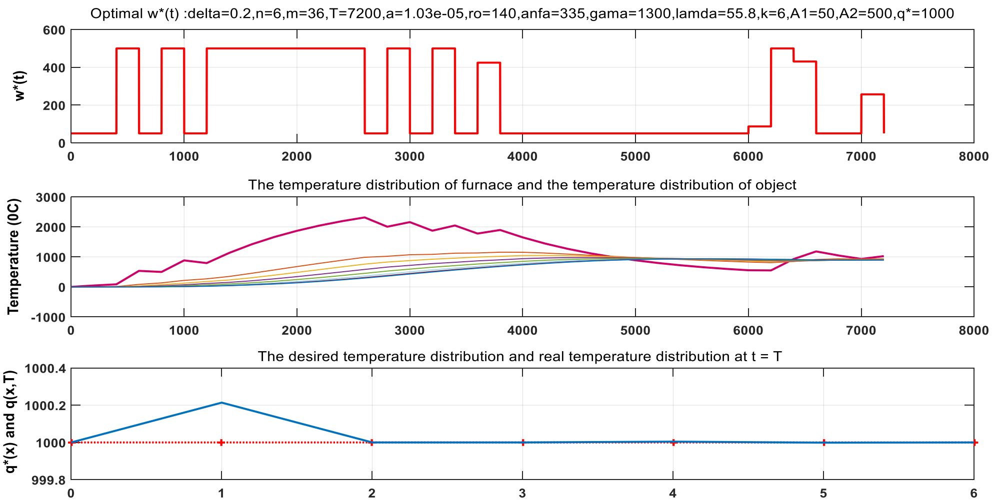

Figure 1: The optimal heating process for a slab of steel

In figure 1, *w*(t)* is the optimal control signal (optimal power) of the furnace; *u(t)* is the temperature of the furnace; *q(x,t)* is temperature distribution of the slab, including the temperature of the two surfaces and the temperature of the inner layers of the slab of steel.

From the figure 1, we can see that at the time $t = T$, the temperature distributions of the layers in a slab of steel $q(x,T)$ are all approximately equal $1000^{0}C$. The largest error in compasison with the desired temperature is about 0.2^{0}C. Therefore, the optimal solution has been testified.

5.2. The simulation for a slab of Samot

- The physical parameters of the object
  - *The heat transfer coefficient* $\alpha = 60$ *(w/m^2. ^{0}C)*
  - *The heat conducting coefficient* $\lambda = 1.995$ *(w/m. ^{0}C)*
  - *The temperature conducting factor a* $= 4.86*10^{-7}$ *(m^2/s)*
  - *The thickness of the object* $\delta = 0.027$ *(m)*
- The parameters of the furnace
  - *The time constant* $\gamma = 1200$ *(s)*
  - *The time delay of the furnace* $\rho = 130$ *(s)*
  - *The static transfer coefficient of the furnace k* $= 0.277$
- The desired temperature distribution $q^*(x) = 500\ ^0C$
- The period of heating time $T = 5400$ (s)
- Limit the temperature of furnace $u(t) \le 800^0$C
- Limit the temperature of slab surface $q(0,t) \le 600^0$C

With these parameters, the coefficient *Bi* is calculated as follows:

$$Bi = \frac{\alpha.\delta}{\lambda} = \frac{60.0,027}{1.995} \approx 0,81 \tag{40}$$

Thus, the slab of Samot is also a thick object because the coefficient Bi is greater than 0.5. Having $6 \le \gamma/\rho < 10$.
To calculate the optimal heating process, we choose $n = 6$ and $m = 36$, too. After the simulation, we have results like in figure 2.

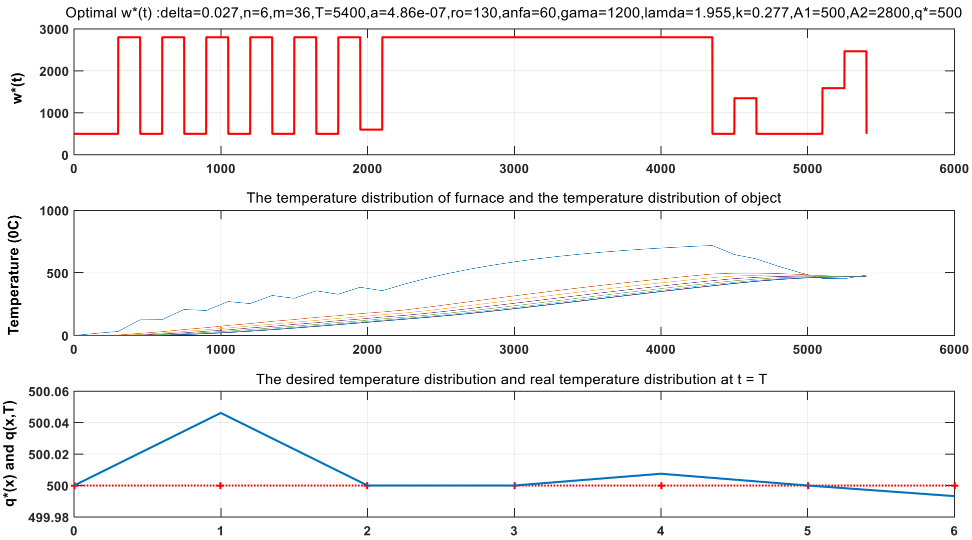

Figure 2: The optimal heating process for a slab of Samot

From the figure 2, we can see that at the time $t = T$, the temperature distributions of the layers in a slab of Samot $q(x,T)$ are all approximately equal 500^0C. The largest error in comparison with the desired temperature is about 0.05^0C. Therefore, the optimal solution has been testified.

6. Conclusions

The paper has offered an approximation method with higher accuracy to replace a delayed object by using first-order Pade approximation model. A relationship between the provided power for the furnace *w(t)* and the temperature distribution of the object *q(x,t)* has been found. Namely, we have solved a system consisting of the partial differential equation type Parabolic with boundary condition type-3 combined with a time-delayed ordinary differential equation. An optimal solution for DPSTD has been defined by using a numerical method. Algorithms and optimal calculating program have been accuracy. Then, we have proceeded to run the simulations on a slab of steel and a slab of Samot in order to test the algorithms once again.

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