International Journal of Computing and Optimization
Vol. 4, 2017, no. 1, 11 - 18
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijco.2017.61024

New Propositions on the Affine-Scaling Interior-Point Algorithm

Douglas Kwasi Boah

Faculty of Mathematical Sciences
Department of Mathematics, University for Development Studies
P. O. Box 24, Navrongo, Ghana

Stephen Boakye Twum

Faculty of Mathematical Sciences
Department of Mathematics, University for Development Studies
P. O. Box 24, Navrongo, Ghana

Abstract

In this paper, two new propositions on the Affine-Scaling Interior Point Algorithm of Linear Programming have been put forward. The first proposition establishes a relationship between the number of iterations of the Affine-scaling interior-point algorithm required to obtain an optimal solution of any Linear Programming problem involving only inequality constraints and the number of decision variables. The second proposition gives a formal statement on the consequence of a selected constant value of zero (0) on the Affine-scaling interior-point algorithm. The propositions are recommended for use by all operations researchers and computer scientists around the world.

Keywords: Interior Point Methods, Affine-Scaling Interior Point Algorithm, Optimal Solution, Linear Programming, Initial Feasible Trial Solution

Introduction

The Simplex Method (SM) remained a popular solution method of practical linear programming (LP) problems, until the development of interior point methods.

Karmarkar (1984) [9] was the pioneer in the field and his Projective Scaling Method was able to compete with the SM as applied to realistic problems. As the name suggests, interior point methods approach an optimal point (which is known to be on the boundary of the feasible set) through a sequence of interior points (Eiselt and Sandblom, 2007 [6]). Unlike the SM, iterates are calculated not on the boundary, but in the interior of the feasible region. Starting with an initial interior point, the method moves through the interior of the feasible set along an improving direction to another interior point. There, a new improving direction is found, along which a move is made to yet another interior point. This process is repeated, resulting in a sequence of interior points which converge to an optimal boundary point. Many different types of interior-point methods for linear programming have been developed. Most of the methods fall under one of the three main categories: the projective and potential reduction method, affine-scaling method and path-following methods (Singh and Singh, 2002 [13]). In this paper, two new propositions on the Affine-Scaling Interior Point Algorithm of LP have been put forward.

Materials and Methods

The Affine-Scaling Interior-Point Algorithm was first introduced by Dikin (1967) [4]. He subsequently published a convergence analysis in Dikin (1974) [5]. Dikin's work went largely unnoticed for many years until Karmakar & Ramakrishnan (1985) [10], Barnes (1986) [3], Vanderbei & Freedman (1986) [16] and Adler et al (1989) [2] rediscovered it as a simple variant of Karmarkar's algorithm. Here, the problem is rescaled in order to make the initial point stay some distance away from any boundary constraint and then restrict the step length, so that the next move will not reach the boundary. The algorithm is as follows.

Given an optimization problem in standard form:
Optimize $Z = c^T x$

Subject to $Ax = b$

$x \geq 0,$

where c, A and b are the cost coefficients, technological coefficients and resource availability respectively, the Affine-Scaling Interior-Point Algorithm is summarized in the following steps:

Step 1: Given the initial trial solution, $x = (x_1\ ,\ x_2\ , \dots\ , x_n)^T$, set

$$D = \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \ldots & 0 \\ 0 & 0 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & x_n \end{bmatrix}$$

Step 2: Calculate $\tilde{A} = AD \;\; and \; \tilde{c} = Dc.$
Step 3: Calculate $P = I - \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} \quad and \; C_p \; = P\tilde{c}$ where P is a projection matrix and C_p is a projected gradient.
Step 4: Identify the negative component of $\boldsymbol{C_p}$ having the largest absolute value, and set $\boldsymbol{v}$ to this absolute value. Then Calculate

$$\tilde{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\theta}{v} C_p, \qquad [1.0]$$

where $0 \; < \theta \; < 1$ (Dikin, 1967) [4]

Step 5: Calculate $x = D\tilde{x}$ as the trial solution for the next iteration starting from step 1.

(If this trial solution is virtually unchanged from the preceding one, then the algorithm has virtually converged to an optimal solution and the algorithm is terminated) [Hillier and Lieberman, 2010] [8]

Several other researchers have done a lot of work on the Affine-Scaling Interior Point Method. Todd (1991) [14] presented 'The affine-scaling direction for linear programming is a limit of projective-scaling directions'. Kim and Nazareth (1994) [11] presented a primal null-space affine-scaling method. Again, Achiya (2001) [1] presented 'Loss and retention of accuracy in affine-scaling method'. Nayak et al (2012) [12] presented an affine-scaling method for solving network flow problems. Furthermore, Hager and Zhang (2014) [7] presented an affine-scaling method for optimisation problems with polyhedral constraints.

Results and Discussions

New Propositions on the Affine-Scaling Interior Point Algorithm

In the Affine-scaling interior-point algorithm (Dikin, 1967) [4] discussed under above, the selected constant, θ in Equation 1.0 is required to be such that $0 < \theta < 1$. Thus according to Dikin (1967) [4], the possible θ values should exclude 0 and 1. The selected constant, θ measures the fraction used of the distance that could be moved before the feasible region is exited (Hillier and Lieberman, 2010) [8]. Dikin (1974) [5] published convergence analysis of the method using θ = 0.5. Tsuchiya and Muramatsu (1995) [15] used θ = 2/3 in their convergence result. Hillier et al (2010) [8] used θ values of 0.5 and 0.9 in their Interactive Operations Research (IOR) software. In this study, investigations into the consequence of θ values of zero (0) and one (1) on the algorithm have been undertaken. Subsequently, it has been observed that, θ values of zero (0) and one (1) give very remarkable results. The observations have led to two propositions on the Affine-scaling interior-point algorithm as follows:

Proposition 1: Given any linear programming problem involving only inequality constraints, the number of iterations of the Affine-scaling interior-point algorithm required to obtain an optimal solution is equal to the number of decision variables, if the selected value of θ is one (1).

Tables 1 (a) and (b) present some computational results of selected practical problems affirming proposition 1. The table specifies the LP problems, selected θ values (with their corresponding number of iterations in brackets) and their optimal solutions using a developed Interior-Point Program based on the Affine-Scaling Interior Point Algorithm which was written in MATLAB.

Table 1 (a): Computational Results of Selected Practical Problems Affirming Proposition 1

LP Problem	θ values and their corresponding number of iterations in brackets	Optimal Solution
Maximize $Z = 6X_1 + 8X_2$ *Subject to* $X_1 + 2X_2 \leq 12$ $X_1 + X_2 \leq 10$ $X_1, X_2 \geq 0$	0.1→ (79) 0.2→ (38) 0.3 → (24) 0.4→ (17) 0.5 → (12) 0.6→ (10) 0.7 → (8) 0.8→ (7) 0.9 → (6) 1.0 → (2)	$Z = 64$ $X_1 = 8$ $X_2 = 2$
Maximize $Z = 2X_1 + 3X_2$ *Subject to* $X_1 + X_2 \geq 350$ $2X_1 + X_2 \leq 600$ $X_1 \geq 125$ $X_1, X_2 \geq 0$	0.1→ (92) 0.2→ (45) 0.3 → (29) 0.4→ (21) 0.5 → (17) 0.6→ (12) 0.7 → (11) 0.8 → (10) 0.9 → (8) 1.0 → (2)	$Z = 1300$ $X_1 = 125$ $X_2 = 350$
Minimize $Z = 3X_1 + 2X_2$ *Subject to* $5X_1 + X_2 \geq 10$ $X_1 + X_2 \geq 6$ $X_1 + 4X_2 \geq 12$ $X_1, X_2 \geq 0$	0.1→ (90) 0.2→ (43) 0.3 → (27) 0.4→ (20) 0.5 → (15) 0.6→ (11) 0.7 → (10) 0.8→ (9) 0.9 → (7) 1.0 → (2)	$Z = 13$ $X_1 = 1$ $X_2 = 5$

Table 1 (a): (Continued): Computational Results of Selected Practical Problems Affirming Proposition 1

Minimize $Z = 20X_1 + 10X_2$ *Subject to* $X_1 + 2X_2 \leq 40$ $3X_1 + X_2 \geq 30$ $4X_1 + 3X_2 \geq 60$ $X_1, X_2 \geq 0$	0.1→ (90) 0.3 → (27) 0.5 → (15) 0.7 → (10) 0.9 → (7)	0.2→ (43) 0.4→ (20) 0.6→ (11) 0.8→ (9) 1.0 → (2)	$Z = 240$ $X_1 = 6$ $X_2 = 12$
Maximize $Z = 16X_1 + 17X_2 + 10X_3$ *Subject to* $X_1 + X_2 + 4X_3 \leq 2000$ $2X_1 + X_2 + X_3 \leq 3600$ $X_1 + 2X_2 + 2X_3 \leq 2400$ $X_1 \leq 30$ $X_1, X_2, X_3 \geq 0$	0.1→ (122) 0.3 → (37) 0.5 → (19) 0.7 → (12) 0.9 → (10)	0.2→ (58) 0.4→ (26) 0.6→ (15) 0.8→ (11) 1.0 → (3)	$Z = 20625$ $X_1 = 30$ $X_2 = 1185$ $X_3 = 0$

Table 1 (b): Computational Results of Selected Practical Problems Affirming Proposition 1

LP Problem	**θ values and their corresponding number of iterations in brackets**		**Optimal Solution**
Minimize $Z = 1.06X_1 + 0.56X_2 + 3.00X_3 + 2703.50X_4 + 4368.23X_5$ *Subject to* $1.06X_1 + 0.015X_3 \geq 729824.87$ $0.56X_2 + 0.649X_3 \geq 1522188.03$ $3.00X_3 \geq 5040.16$ $2703.50X_4 \geq 162210.06$ $4368.23X_5 \geq 17472.92$ $X_1, X_2, X_3, X_4, X_5 \geq 0$	0.1→ (130) 0.3 → (42) 0.5 → (22) 0.7 → (15) 0.9 → (13)	0.2→ (62) 0.4→ (29) 0.6→ (16) 0.8 → (14) 1.0 → (5)	$Z = 2435620.485$ $X_1 = 688490.254$ $X_2 = 2716245.849$ $X_3 = 1680.053$ $X_4 = 60.000$ $X_5 = 4.000$
Minimize $Z = 2.03X_1 + 0.56X_2 + 2.93X_3 + 1543.85X_4 + 1494.14X_5$ *Subject to* $2.03X_1 + 0.015X_3 \geq 3604.90$ $0.56X_2 + 0.633X_3 \geq 430264.03$ $2.93X_3 \geq 750.50$ $1543.85X_4 \geq 26245.39$ $1494.14X_5 \geq 5976.56$ $X_1, X_2, X_3, X_4, X_5 \geq 0$	0.1→ (130) 0.3 → (42) 0.5 → (22) 0.7 → (15) 0.9 → (13)	0.2→ (62) 0.4→ (29) 0.6→ (16) 0.8 → (14) 1.0 → (5)	$Z = 466675.399$ $X_1 = 1773.920$ $X_2 = 768039.091$ $X_3 = 256.143$ $X_4 = 17.000$ $X_5 = 4.000$

It is seen from Tables 1 (a) and (b) that, the number of iterations decreases as θ values increase. Also, in each of the LP problems, the number of iterations required to obtain an optimal solution is equal to the number of the decision variables if the selected θ value is one (1) and this clearly affirms Proposition 1.

Proposition 2: Given any linear programming problem, the initial feasible trial solution will be the final solution, if the selected value of θ in the Affine-scaling interior-point algorithm is zero (0).

Proof

Given an optimization problem in the standard form:

Optimize $Z = c^T x$

Subject to $Ax = b$

$x \geq 0,$

where c, A and b are the usual parameters respectively.
Given the initial trial solution, $x = (x_1 , \ x_2 , \dots , x_n)^T$, we set

$$D = \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & x_n \end{bmatrix}$$

We then calculate $\tilde{A} = AD \ \ and \ \tilde{c} = Dc.$
Again, we calculate $P = I - \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} \quad and \ C_p \ = P\tilde{c}$ where P is a projection matrix and C_p is a projected gradient.

Next, we identify the negative component of C_p having the largest absolute value, and set $\boldsymbol{v}$ to this absolute value and then calculate

$$\tilde{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{\theta}{v} C_p.$$

Since θ = 0, $\tilde{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $x = D\tilde{x} = (x_1 , \ x_2 , \dots , x_n)^T$ is the trial solution

for the next iteration.
Since this trial solution is the same as the preceding one, we terminate the algorithm. We therefore conclude that, the final solution is the same as the initial feasible trial solution. This completes the proof of proposition 2.

It must be noted here that, although Proposition 2 might look trivial, it categorically gives a formal statement on the consequence of $\theta = 0$ on the Affine-scaling interior-point algorithm which was not specified in Dikin (1967) [4].

Conclusion and Recommendations

Two new propositions on the Affine-Scaling Interior Point Algorithm of LP have been put forward. Computational results of selected practical problems affirming the proposition have been provided. Clearly, given any LP problem involving only inequality constraints, the relationship between the number of iterations of the Affine-scaling interior-point algorithm required to obtain an optimal solution and the number of decision variables, if the selected value of θ is one (1) has been established. The second proposition has been proved theoretically or rigorously. Unequivocally, a formal statement on the consequence of $\theta = 0$ on the Affine-scaling interior-point algorithm has been given. The propositions are recommended for use by all operations researchers and computer scientists around the world.

References

[1] D. Achiya, Loss and retention of accuracy in affine scaling methods, *Optimisation Methods and Software*, **15** (2001), no. 2, 121-151. https://doi.org/10.1080/10556780108805814

[2] I. Adler, M. Resende, G. Veiga and N. Karmarkar, An implementation of Karmarkar's algorithm for linear programming, *Mathematical Programming*, **44** (1989), 297-335. https://doi.org/10.1007/bf01587095

[3] E. R. Barnes, A variation on Karmarkar's algorithm for solving linear programming problems, *Mathematical Programming*, **36** (1986), 174-182. https://doi.org/10.1007/bf02592024

[4] I. I. Dikin, Iterative solution of problems of linear and quadratic programming, *Soviet Mathematics Doklady*, **8** (1967), 674-675.

[5] I. I. Dikin, On the speed of an iterative process, *Upravlyaemye Sistemi*, **12** (1974), 54-60.

[6] H. A. Eiselt and C. L. Sandblom, *Linear Programming and its Applications*, Springer-Verlag Berlin Heidelberg, 2007. https://doi.org/10.1007/978-3-540-73671-4

[7] W. W. Hager and H. Zhang, An affine scaling method for optimisation problems with polyhedral constraints, *Computational Optimisation and Applications*, **59** (2014), 163-183. https://doi.org/10.1007/s10589-013-9535-x

[8] F. S. Hillier and G. J. Lieberman, *Introduction to Operations Research*, Ninth Edition, 2010.

[9] N. Karmarkar, A new Polynomial-time algorithm for linear programming, *Combinatorica*, **4** (1984), 373-395. https://doi.org/10.1007/bf02579150

[10] N. Karmarkar and U. Ramakrishnan, *Further Developments in the New Polynomial-Time Algorithm for Linear Programming*, Talk given at ORSA/TIMS National Meeting, Boston, MA, 1985.

[11] K. Kim and J. L. Nazareth, A primal null-space affine-scaling method, *Transactions on Mathematical Software*, **20** (1994), no. 3, 373-392. https://doi.org/10.1145/192115.192153

[12] R. K. Nayak, M. P. Biswal and S. Padhy, An affine scaling method for solving network flow problems, *Journal of Discrete Mathematical Sciences and Cryptography*, **15** (2012), 13-29. https://doi.org/10.1080/09720529.2012.10698361

[13] J. N. Singh and D. Singh, Interior-point methods for linear programming: a review, *International Journal of Mathematical Education in Science and Technology*, **33** (2002), no. 3, 405-423. https://doi.org/10.1080/00207390110119934

[14] M. J. Todd, The affine-scaling direction for linear programming is a limit of projective-scaling directions, *Linear Algebra and its Applications*, **152** (1991), 93-105. https://doi.org/10.1016/0024-3795(91)90268-2

[15] T. Tsuchiya and M. Muramatsu, Global convergence of a long-step affine scaling algorithm for degenerate linear programming problems, *SIAM Journal on Optimization,* **5** (1995), no. 3, 525-551. https://doi.org/10.1137/0805027

[16] R. J. Vanderbei, M. S. Meketon and B. A. Freedman, A modification of Karmarkar's linear programming algorithm, *Algorithmica*, **1** (1986), 395-407. https://doi.org/10.1007/bf01840454

Received: November 11, 2016; Published: December 27, 2016