On a Family of Symmetric Numerical Semigroups with Embedding Dimension Three

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Abstract

In this paper we consider numerical semigroups generated by the positive integers \( n, 2(n + 1), \) and \( 3(n + 1) \) when \( n \geq 4 \). We establish formulas for the Apéry set, Frobenius number and genus and prove that these numerical semigroups are always symmetric. We then show how to identify all the isolated gaps and construct some perfect numerical semigroups of embedding dimension five.

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1 Introduction

Let \( \mathbb{N}_0 \) denote the set of all non-negative integers. A numerical semigroup is an additive submonoid \( S \) of \( \mathbb{N}_0 \) such that \( \mathbb{N}_0 \setminus S \) is finite (see [11] for a comprehensive reference on numerical semigroups). The gaps of \( S \) are the elements of the set \( G(S) = \mathbb{N}_0 \setminus S \). The cardinality of \( G(S) \) is called the genus of \( S \) and is denoted by \( g(S) \). The largest element of \( G(S) \) is called the Frobenius number of \( S \) and is denoted by \( F(S) \). The determination of \( F(S) \) for a given
A numerical semigroup $S$ is symmetric if $F(S) - x \in S$. It is straightforward to show that a numerical semigroup $S$ is symmetric if and only if $g(S) = (F(S) + 1)/2$ (see, for example, Corollary 4.5 in [11]). Symmetric numerical semigroups are of interest in ring theory and algebraic geometry. In [5] it was shown that a one-dimensional analytically irreducible Noetherian local ring is Gorenstein if and only if its associated value semigroup is symmetric (see also [1] for similar results).

Let $S$ be a numerical semigroup and let $A \subseteq S$. We say that $A$ is a system of generators of $S$ if, for every $s \in S$, there exist $a_1, \ldots, a_k \in A$ and $n_1, \ldots, n_k \in \mathbb{N}_0$ such that $s = n_1 a_1 + \cdots + n_k a_k$. In this case, we use the notation $S = \langle A \rangle$. It is known that if $A$ is a nonempty subset of $\mathbb{N}_0$, then $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$ (see, for example, Lemma 2.1 of [11]). It is also known that every numerical semigroup can be generated by finitely-many elements and that every numerical semigroup has a unique minimal system of generators (see Theorem 2.7 of [11]). The cardinality of the minimal system of generators is called the embedding dimension of $S$. Numerical semigroups with embedding dimension two are thoroughly understood, but those with embedding dimension greater than two are still being studied (see, for example, [2], [3], [4], [7], and [12]).

Let $S$ be a numerical semigroup and let $n$ be one of its nonzero elements. The Apéry set of $n$ in $S$ is the set $\text{Ap}(S, n) = \{s \in S|s - n \notin S\}$. Apéry sets are among the most versatile tools for studying numerical semigroups. It is known that $\text{Ap}(S, n)$ is a complete system of residues modulo $n$ and that $\text{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n - 1)\}$ where $w(i)$ is the least element of $S$ congruent to $i$ modulo $n$, for all $i \in \{0, \ldots, n - 1\}$ (see, for example, Lemma 2.4 of [11]). In [12], Selmer established the following useful formulas in terms of the Apéry set (the proof of each can also be found as Proposition 2.12 in [11]):

$$F(S) = (\max \text{Ap}(S, n)) - n \quad (1)$$

$$g(S) = \frac{1}{n}\left(\sum_{w \in \text{Ap}(S, n)} w\right) - \frac{n - 1}{2} \quad (2)$$

A gap $x$ of a numerical semigroup $S$ is an isolated gap if $x - 1, x + 1 \in S$. We will let $I(S)$ denote the set of all isolated gaps of $S$. A perfect numerical semigroup is one in which $I(S) = \emptyset$. Perfect numerical semigroups have been explored in [8], [9], [13], and [14].

In this paper, we will investigate some properties of a family of numerical semigroups with embedding dimension three. We will completely describe their Apéry sets, then use Selmer’s formulas to calculate the Frobenius number and genus of each and show that they are all symmetric. We will then show how to
identify all of the isolated gaps of each. This will allow us to easily construct some perfect numerical semigroups with embedding dimension 5.

2 Main Results

Let \( n \in \mathbb{N} \) and note that \( 1 = -n - (2n + 2) + (3n + 3) \). Since 1 can be expressed as an integer combination of \( n, 2n + 2, \) and \( 3n + 3 \), it follows that \( \gcd(n, 2n + 2, 3n + 3) = 1 \) and so \( S = \langle n, 2n + 2, 3n + 3 \rangle \) is a numerical semigroup. Note that when \( n = 2 \) or 3, \( S \) has embedding dimension two. It is known that every numerical semigroup of embedding dimension two is symmetric and formulas for the genus, Frobenius number, and Apéry set in terms of the minimal generators are also known (see, for example, Proposition 2.13 and Corollary 4.7 of [11]). Therefore, we will restrict our attention to those cases when \( n \geq 4 \).

**Theorem 2.1.** Let \( n \geq 4 \) and let \( S = \langle n, 2(n+1), 3(n+1) \rangle \). Then \( \text{Ap}(S, n) = \{0, 2(n+1), 3(n+1), \ldots, (n-1)(n+1), (n+1)(n+1)\} \).

**Proof.** Let \( \text{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\} \) where \( w(i) \) is the least element of \( S \) congruent to \( i \) modulo \( n \), for each \( i \in \{0, \ldots, n-1\} \).

We claim that \( w(1) = (n+1)^2 \) and \( w(i) = i(n+1) \) for each \( i \in \{2, \ldots, n-1\} \). Note that since \( 2(n+1), 3(n+1) \in S \), it follows that \( k(n+1) \in S \) for all \( k \geq 2 \). Moreover, \( k(n+1) \equiv k \pmod{n} \) for each \( k \in \{2, \ldots, n-1, n+1\} \). Therefore, we need only show that \( k(n+1) - n \notin S \) for each \( k \in \{2, \ldots, n-1, n+1\} \). We prove by contradiction. Suppose that \( k \in \{2, \ldots, n-1, n+1\} \) and \( k(n+1) - n \in S \). Then \( k(n+1) - n = an + b(n+1) \) for some \( a, b \geq 0 \). This implies that \( b \geq 2 \) (since \( b \equiv k \pmod{n} \)). However, it also implies that \( k(n+1) = (a+1)n + b(n+1) \) and so \( b < k \). Hence, since \( b \equiv k \pmod{n} \), it follows that \( b \leq k - n \leq n + 1 - n = 1 \) (a contradiction). \( \square \)

Using this result and Selmer’s formulas, we may establish the following.

**Corollary 2.2.** Let \( n \geq 4 \) and let \( S = \langle n, 2(n+1), 3(n+1) \rangle \). Then \( F(S) = n^2 + n + 1 \) and \( g(S) = (n^2 + n + 2)/2 \). In particular, \( S \) is symmetric.

**Proof.** Recall from Selmer’s formula (1) that \( F(S) = (\max \text{Ap}(S, n)) - n \). Hence, by Theorem 2.1, \( F(S) = (n+1)^2 - n = n^2 + n + 1 \).

Recall also from Selmer’s formula (2) that \( g(S) = (\sum_{w \in \text{Ap}(S, n)} w) / n - (n-1)/2 \).
By Theorem 2.1,
\[
\sum_{w \in Ap(S,n)} w = \left( \sum_{k=2}^{n+1} k(n+1) \right) - n(n+1) = (n+1) \left( \sum_{k=2}^{n+1} k \right) - n(n+1) = (n+1) \left( \frac{n+1(n+2)}{2} - 1 \right) - n(n+1) = \frac{n(n+1)^2}{2}
\]

Thus, \( g(S) = \frac{n(n+1)^2}{2n} - \frac{n-1}{2} = \frac{n^2 + n + 2}{2} \). Finally, note that \( S \) is symmetric because \( g(S) = (F(S)+1)/2 \). \( \square \)

When \( n \geq 4 \), a numerical semigroup of the form \( S = \langle n, 2(n+1), 3(n+1) \rangle \) can never be a perfect numerical semigroup. To see why, note that if \( I(S) = \emptyset \), then \( F(S) \not\subseteq I(S) \) and so \( F(S) - 1 \not\subseteq S \). But, since \( S \) is symmetric, it now follows that \( F(S) - (F(S)-1) = 1 \in S \) (which cannot be the case if \( n > 1 \)). Therefore, if \( S = \langle n, 2(n+1), 3(n+1) \rangle \), then \( I(S) \neq \emptyset \). We now show how to identify all of the isolated gaps of these numerical semigroups.

**Theorem 2.3.** Let \( n \geq 4 \) and let \( S = \langle n, 2(n+1), 3(n+1) \rangle \). Then \( I(S) = \{2n+1, 3n+1, \ldots, (n+1)n+1, n^2 - n - 1\} \).

**Proof.** Let \( U = \{2n+1, 3n+1, \ldots, (n+1)n+1, n^2 - n - 1\} \). We will use Theorem 2.1 repeatedly to prove that \( U = I(S) \).

First, we prove \( U \subseteq I(S) \). Note that \( n^2 - n - 1 = (n-1)(n+1) - n = w(n-1) - n \not\in S \). Moreover, \( n^2 - n - 1 + 1 \in n\mathbb{N}_0 \subseteq S \) and \( n^2 - n - 1 - 1 = (n-2)(n+1) = w(n-2) \in S \). Hence, \( n^2 - n - 1 \in I(S) \). We now show that \( kn + 1 \in I(S) \) for each \( k \in \{2, \ldots, n+1\} \). Note \( kn + 1 \equiv 1 \pmod{n} \) and \( kn + 1 \leq (n+1)n+1 < (n+1)^2 = w(1) \). Hence, \( kn + 1 \not\in S \). Clearly \( kn + 1 - 1 \in n\mathbb{N}_0 \subseteq S \). Moreover, \( kn + 1 + 1 \geq 2n + 2 = w(2) \). Since \( kn + 1 + 1 \equiv 2 \pmod{n} \), \( kn + 1 + 1 \in S \) and so \( kn + 1 \in I(S) \).

Next, we prove \( I(S) \subseteq U \). To do this, we will show (1) that every element of \( I(S) \) must be congruent to 1 or \( n - 1 \) modulo \( n \), (2) that the only elements of \( I(S) \) congruent to 1 modulo \( n \) are of the form \( kn + 1 \) where \( 2 \leq k \leq n+1 \), and (3) that the only element of \( I(S) \) congruent to \( n-1 \) modulo \( n \) is \( n^2 - n - 1 \).

First, we show that if \( x \in G(S) \) and \( x \equiv i \pmod{n} \), with \( i \in \{2, \ldots, n-2\} \), then \( x + 1 \not\in S \) (and so \( x \not\in I(S) \)). Assume \( x \in G(S) \) and \( x \equiv i \pmod{n} \), with \( i \in \{2, \ldots, n-2\} \). Then \( x < w(i) = i(n+1) \). Hence, \( x + 1 \equiv i+1 \pmod{n} \) and \( x + 1 < i(n+1) + 1 < (i+1)(n+1) = w(i+1) \) and so \( x + 1 \not\in S \). This proves that every element of \( I(S) \) must be congruent to 1 or \( n - 1 \) modulo \( n \).

We now prove that the only elements of \( I(S) \) congruent to 1 modulo \( n \) are of the form \( kn + 1 \) where \( 2 \leq k \leq n+1 \). Note that if \( k > n+1 \),
and $kn + 1 > (n + 1)n + 1 = F(S)$ and so $kn + 1 \in S$. If $k < 2$, then $kn + 1 + 1 < 2(n + 1) = w(2)$. Since $kn + 2 \equiv 2 \pmod{n}$ and $kn + 2 < w(2)$, $kn + 2 \notin S$ and so $kn + 1 \notin I(S)$.

Finally, we show that $n^2 - n - 1$ is the only isolated gap congruent to $n - 1$ modulo $n$. Suppose that $x \in I(S)$ and $x \equiv n - 1 \pmod{n}$. If $x > n^2 - n - 1$, then $x \geq n^2 - 1 = (n - 1)(n + 1) = w(n - 1)$ and so $x \in S$ (a contradiction). Hence, $x \leq n^2 - n - 1$. If $x < n^2 - n - 1$, then $x - 1 < n^2 - n - 2 = (n - 2)(n + 1) = w(n - 2)$. Since $x - 1 \equiv n - 2 \pmod{n}$, this shows that $x - 1 \notin S$ and so $x \notin I(S)$. 

The following useful fact about the isolated gaps of a numerical semigroup appears as Proposition 2.1 in [14].

**Lemma 2.4.** Let $S$ be a numerical semigroup. If $x, y \in I(S)$ and $s \in S$, then $x + y \in S$ and $x + s \subseteq I(S) \in S$.

**Proof.** By definition, $x + 1, y - 1 \in S$ and so $x + y = x + 1 + y - 1 \in S$. Note that if $x + s \in G(S)$, then $x + s \pm 1 \in S$ (since $x \in I(S)$) and so $x + s \in I(S)$. 

Finally, we use these results to construct some perfect numerical semigroups.

**Theorem 2.5.** Let $n \geq 4$. Then $T = \langle n, 2(n + 1), 3(n + 1), 2n + 1, n^2 - n - 1 \rangle$ is a perfect numerical semigroup.

**Proof.** Suppose there exists $x \in I(T)$. Let $S = \langle n, 2(n + 1), 3(n + 1) \rangle$. By Theorem 2.3, $I(S) = \{2n + 1, 3n + 1, \ldots, (n + 1)n + 1, n^2 - n - 1\}$. Note that $S, I(S) \subseteq T$ and so $x$ cannot be an isolated gap of $S$. Therefore, $x - 1 \in G(S)$ or $x + 1 \in G(S)$.

We will show that $x - 1 \in G(S)$ leads to a contradiction (a similar proof shows that $x + 1 \in G(S)$ leads to the same contradiction). Assume that $x - 1 \in G(S)$. Since $x \in I(T)$, $x - 1 \in T$ and so $x - 1 = s + a(2n + 1) + b(n^2 - n - 1)$ for some $s \in S$ and some $a, b \geq 0$. By Lemma 2.4, it follows that $x - 1 \in I(S) \cup S$. Since $x - 1 \notin S$, we must have that $x - 1 \in I(S)$. Hence, $x - 1 + 1 = x \in S$. But since $S \subseteq T$, we now have that $x \in T$, a contradiction. Therefore, $I(T) = \emptyset$ and $T$ is a perfect numerical semigroup.

When $n = 4$, the resulting numerical semigroup $T$ is perfect, but has embedding dimension 4. When $n \geq 5$, perfect numerical semigroups of embedding dimension 5 can be obtained.

**Example 2.6.** Let $n = 5$. Then

$S = \langle 5, 12, 18 \rangle = \{0, 5, 10, 12, 15, 17, 18, 20, 22, 23, 24, 25, 27, 28, 29, 30, 32 \rightarrow \}$
is symmetric with $F(S) = 31$ and $g(S) = 16$. Moreover,

$$T = \langle 5, 12, 18, 11, 19 \rangle = \{0, 5, 10, 11, 12, 15 \rightarrow \}$$

is a perfect numerical semigroup.

References


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