Abstract

In this paper, as a generalization of the essential topology of dcpos, the concept of Z-essential topology on Z-complete posets is introduced. Basic properties of the Z-essential topology and relations with other intrinsic topologies are explored. Via the Z-essential topology, we obtain properties and characterizations of Z-bases of strongly Z-continuous posets.

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1 Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [6]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or continuous domains) was introduced and extensively studied (see [2], [3]). Lawson in [3] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. In order to provide a topological interpretation of bases of continuous dcpos, Rusu and Ciobanu in [5] introduced the concepts of the essential topology and the density topology of dcpos and proved that bases are just dense sets in the density topology of continuous dcpos.

The use of $Z$-subset systems to study the common properties of classes of posets such as completely distributive lattices and continuous dcpos first appeared in [9]. Responding to the suggestion in [9], $Z$-continuous posets and their properties were introduced and studied in [1, 7]. The $Z$-subset system is a generalization of the directed set system. Since the concept was defined, many theorems in domain theory have been given the corresponding conclusions with respect to $Z$-subset system.

In this paper, we introduce the concept of $Z$-essential topology which is a generalization of the notion of the essential topology on dcpos. We investigate properties of the $Z$-essential topology and relations with other intrinsic topologies. Via the $Z$-essential topology, we obtain several properties and characterizations of $Z$-bases of strongly $Z$-continuous posets.

2 Preliminaries

We quickly recall some basic notions and results (see [2] and [7]).

Let $(P, \leq)$ be a poset. We use $2^P$ or $\mathcal{P}(P)$ to denote the powerset of $P$. A principal ideal (resp., principal filter) is a set of the form $\downarrow x = \{y \in P \mid y \leq x\}$ (resp., $\uparrow x = \{y \in P \mid x \leq y\}$). For $A \subseteq P$, we write $\downarrow A = \{y \in P \mid \exists x \in A, y \leq x\}$, $\uparrow A = \{y \in P \mid \exists x \in A, x \leq y\}$. A subset $A$ is a lower set (resp., an upper set) if $A = \downarrow A$ (resp., $A = \uparrow A$). The supremum of $A$ is denoted by $\bigvee A$ or $\sup A$. The infimum of $A$ is denoted by $\bigwedge A$ or $\inf A$. A nonempty subset $D$ of $P$ is directed if every finite subset of $D$ has an upper bound in $D$. A poset $P$ is a directed complete partially ordered set (dcpo, for short) if every directed subset of $P$ has a supremum.

The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the lower topology (resp., upper topology) and is denoted by $\omega(P)$ (resp., $\nu(P)$). The topology of all upper sets (resp., lower sets) is called the Alexandrov topology (resp., the dual Alexandrov topology) and is denoted by $\alpha(P)$ (resp., $\alpha^*(P)$).
For exists, the relation $y$ below $y$ with respect to $Z$, written $x \ll_Z y$, if for any $S \in Z(P)$ such that $\bigvee S$ exists, the relation $y \ll S y$ implies that there exists $s \in S$ such that $x \ll s$. For $x \in P$, we write $\downarrow_Z x = \{ u \in P \mid u \ll_Z x \}$ and $\uparrow_Z x = \{ u \in P \mid x \ll_Z u \}$.

**Definition 2.1.** (see [7]) Let $\textbf{Poset}$ be the category of all posets with monotone maps as morphisms and $\textbf{Set}$ be the category of sets with mappings as morphisms. A subset system on $\textbf{Poset}$ is a functor $Z : \textbf{Poset} \to \textbf{Set}$ satisfying the following conditions:

(i) For any poset $P$, $Z(P) \subseteq 2^P$;

(ii) If $P$ and $Q$ are two posets and $f : P \to Q$ a monotone mapping, then for any $A \in Z(P)$, $Z(f)(A) = f(A) \in Z(Q)$;

(iii) $Z(P)$ contains a nonempty, nonsingleton set for some poset $P$.

**Remark 2.2.** (see [7]) (i) For any poset $P$ and $p \in P$, we have $\{ p \} \in Z(P)$.

(ii) If $P$ is a poset and $Q \subseteq P$, then $Z(Q) \subseteq Z(P)$.

(iii) For each $P$ and each $\{ x, y \} \subseteq P$ such that $x \leq y$, we have $\{ x, y \} \in Z(P)$.

(iv) The family of all directed subsets of $P$, denoted by $Z(P) = \emptyset(P)$, and the family of all subsets of $P$, denoted by $Z(P) = \emptyset(P) = 2^P$, are both subset systems.

**Definition 2.3.** (see [7]) (i) A poset $P$ is called $Z$-complete if for each $S \in Z(P)$, $\bigvee S$ exists.

(ii) A subset $I$ of a poset $P$ is said to be a $Z$-ideal if it is a lower set generated by some $S \in Z(P)$. We shall denote by $I_Z(P)$ the set of all $Z$-ideals of $P$, ordered by inclusion.

**Definition 2.4.** (see [8]) For a poset $P$, let $\sigma_Z(P)$ denote the set of all subsets $U$ of $P$ satisfying the following conditions: (i) $U = \uparrow U$, and (ii) whenever $\sup S$ is in $U$ for some $S \in Z(P)$, then there exists $s \in S$ such that $s \in U$. Let $\sigma_Z^*(P) = \{ F \subseteq P \mid P \setminus F \in \sigma_Z(P) \}$.

**Proposition 2.5.** Let $P$ be a poset. Then

1. $F \in \sigma_Z^*(P)$ iff $F = \downarrow F$ and $\bigvee S \in F$ for any $S \in Z(P)$ with $S \subseteq F$ whenever $\forall S$ exists;

2. $P \in \sigma_Z^*(P)$;

3. If $P$ has a bottom and $\emptyset \in Z(P)$, then $\emptyset \notin \sigma_Z^*(P)$;

4. $\forall x \in P$, $\downarrow x \in \sigma_Z^*(P)$;

5. For any $\{ F_a \}_{a \in \Gamma} \subseteq \sigma_Z^*(P)$, $\bigcap_{a \in \Gamma} F_a \in \sigma_Z^*(P)$ whenever $\cap_{a \in \Gamma} F_a \neq \emptyset$. If $P$ has no bottom, then $\cap_{a \in \Gamma} F_a \in \sigma_Z^*(P)$.

**Proof.** Straightforward. \hfill \Box

**Remark 2.6.** By Proposition 2.5, $\sigma_Z(P)$ need not be a topology for a poset $P$.

**Definition 2.7.** (see [7]) Let $P$ be a poset and $x, y \in P$. We say that $x$ is way-below $y$ with respect to $Z$, written $x \ll_Z y$, if for any $S \in Z(P)$ such that $\bigvee S$ exists, the relation $y \ll S y$ implies that there exists $s \in S$ such that $x \ll s$. For $x \in P$, we write $\downarrow_Z x = \{ u \in P \mid u \ll_Z x \}$ and $\uparrow_Z x = \{ u \in P \mid x \ll_Z u \}$.
The following proposition of the relation \( \ll_Z \) can be immediately verified.

**Proposition 2.8.** Let \( P \) be a poset. Then for all \( x, y, u, v \in P \):

1. \( x \ll_Z y \implies x \leq y \);
2. \( u \leq x \ll_Z y \leq v \implies u \ll_Z v \);
3. If \( P \) has a bottom \( \bot \) and \( \emptyset \notin Z(P) \), then \( \bot \ll_Z x \) for every \( x \in P \). If \( P \) has a bottom \( \bot \) and \( \emptyset \in Z(P) \), then \( \bot \ll_Z x \) for every \( x \in P \setminus \{ \bot \} \), but \( \bot \ll_Z \bot \) does not hold.

**Definition 2.9.** (see [7]) A \( Z \)-complete poset \( P \) is \( Z \)-continuous if for every \( x \in P \), \( 
abla_Z x \in I_Z(P) \) and \( x = \bigvee \downarrow_Z x \). A \( Z \)-continuous poset \( P \) is called strongly \( Z \)-continuous if \( Z \)-continuous if and only if for every \( x \in P \), there exists \( S \in Z(B) \) such that \( S \subseteq \downarrow_Z x \) and \( x = \bigvee S \).

**Lemma 3.2.** Let \( P \) be a \( Z \)-complete poset and \( B \subseteq P \). Then

1. \( P \) is \( Z \)-continuous if and only if for every \( p \in P \), there is a set \( S_p \in Z(P) \) such that \( S_p \subseteq \downarrow_Z x \) and \( p = \bigvee S_p \);
2. \( B \) is a \( Z \)-basis of \( P \) if and only if for each \( x \in P \), there exists \( S \in Z(B) \) such that \( S \subseteq \downarrow_Z x \) and \( x = \bigvee S \).

**Proof.** (1) \( \implies \): If \( P \) is \( Z \)-continuous, then for every \( x \in P \), \( 
abla_Z x \in I_Z(P) \) and \( x = \bigvee \downarrow_Z x \). Thus there is \( S \in Z(P) \) such that \( \downarrow S = \downarrow_Z x \). Showing \( S \subseteq \downarrow S = \downarrow_Z x \) and \( \bigvee S = \bigvee (\downarrow S) = \bigvee \downarrow_Z x = x \).

\( \iff \): If for every \( p \in P \), there is a set \( S_p \in Z(P) \) such that \( S_p \subseteq \downarrow_Z p \) and \( p = \bigvee S_p \), then \( \downarrow_Z p = \downarrow S_p \in I_Z(P) \) and \( p = \bigvee \downarrow_Z p \). By Definition 2.9, \( P \) is \( Z \)-continuous.

(2) \( \implies \): If \( B \) is a \( Z \)-basis of \( P \), then for each \( x \in P \), \( \downarrow_Z x \cap B \in I_Z(B) \) and \( x = \bigvee (\downarrow_Z x \cap B) \). So there is \( S \in Z(B) \) such that \( \downarrow_B S = \downarrow_Z x \cap B \), where \( \downarrow_B S = \{ b \in B \mid \exists s \in S, b \leq s \} \). Clearly, \( S \subseteq \downarrow_B S = \downarrow_Z x \cap B \) and \( \bigvee S = \bigvee (\downarrow_Z x \cap B) = x \).

\( \iff \): If for each \( x \in P \), there exists \( S \in Z(B) \) such that \( S \subseteq \downarrow_Z x \) and \( x = \bigvee S \), then \( \downarrow_Z x \cap B = \downarrow_B S \in I_Z(B) \) and \( x = \bigvee (\downarrow_Z x \cap B) \). By Definition 3.1, \( B \) is a \( Z \)-basis of \( P \).
Proposition 3.3. A Z-complete poset $P$ has a Z-basis if and only if it is Z-continuous.

Proof. Follows immediately from Remark 2.2(ii) and Lemma 3.2. \qed

Proposition 3.4. Let $P$ be a strongly Z-continuous poset. Then for all $x \in P$, the set $\uparrow^Z x \in \sigma_Z(P)$ and for all $U \in \sigma_Z(P)$, $U = \bigcup\{\uparrow^Z y \mid y \in U\}$.

Proof. For all $x \in P$, it follows from by Proposition 2.8(2) that the set $\uparrow^Z x$ is an upper set. For all $S \in Z(P)$ with $\bigvee S \in \uparrow^Z x$, then by the interpolation property of the relation $\ll_Z$ on strongly Z-continuous posets, there is $t \in P$ such that $x \ll_Z t \ll_Z \bigvee S$. So, there exists $s \in S$ such that $x \ll_Z t \leq s$. This shows that $S \cap \uparrow^Z x \neq \emptyset$. By Definition 2.4, $\uparrow^Z x \in \sigma_Z(P)$. Let $U \in \sigma_Z(P)$.

It is clear that $\bigcup\{\uparrow^Z y \mid y \in U\} \subseteq U$. Conversely, for all $u \in U$, it follows from the Z-continuity of $P$ and Lemma 3.2 that there is a set $S_u \in Z(P)$ such that $S_u \subseteq \downarrow^Z u$, $u = \bigvee S_u$ and $S_u \cap U \neq \emptyset$. Hence, we have $\downarrow^Z u \cap U \neq \emptyset$.

Pick $a \in \downarrow^Z u \cap U$. Then $u \in \uparrow^Z a \subseteq \bigcup\{\uparrow^Z y \mid y \in U\}$. This shows that $U \subseteq \bigcup\{\uparrow^Z y \mid y \in U\}$ for all $U \in \sigma_Z(P)$. \qed

Lemma 3.5. Let $P$ be a Z-complete poset. Let $x, y \in P$ with $x \not< y$. If there exists $u \in P$ and $U \in \sigma_Z(P)$ such that $x \in U$, $u \not< y$ and $\uparrow u \cap (P \setminus U) = P$, then $u \ll_Z x$.

Proof. Let $x, y \in P$ with $x \not< y$. Suppose that there is $u \in P$ and $U \in \sigma_Z(P)$ such that $x \in U$, $u \not< y$ and $\uparrow u \cap (P \setminus U) = P$. For any $S \in Z(P)$ with $\bigvee S \geq x$, assume that $\uparrow u \cap S = \emptyset$. Then $S \subseteq P \setminus \uparrow u \subseteq P \setminus U$. By proposition 2.6, we have $\bigvee S \in P \setminus U$ and thus $x \in P \setminus U$, a contradiction to $x \in U$. Therefore, $\uparrow u \cap A \neq \emptyset$ and $u \ll_Z x$. \qed

Lemma 3.6. Let $P$ be a Z-complete poset and $B \subseteq P$ such that for all $x \in P$, $\downarrow^Z x \cap B \in I_Z(B)$. If for any $x, y \in P$ with $x \not< y$, there exist $b \in B$ and $U \in \sigma_Z(P)$ such that $x \in U$, $b \not< y$ and $\uparrow b \cup (P \setminus U) = P$, then $B$ is a Z-basis for $P$.

Proof. Let $x$ be an upper bound of $B$. Assume that $x \not< z$. Then there exists $b_1 \in B$ and $V \in \sigma_Z(P)$ such that $x \in V$, $b_1 \not< z$ and $\uparrow b_1 \cup (P \setminus V) = P$. By Lemma 3.5, $b_0 \ll_Z x$ and hence the set $\downarrow^Z x \cap B$ is nonempty. By Proposition 2.8(1), $x$ is an upper bound of the set $\downarrow^Z x \cap B$. Let $z$ be any upper bound of $\downarrow^Z x \cap B$. Assume that $x \not< z$. Then there exists $b_1 \in B$ and $V \in \sigma_Z(P)$ such that $x \in V$, $b_1 \not< z$ and $\uparrow b_1 \cup (P \setminus V) = P$. By Lemma 3.5, we have $b_1 \ll_Z b_0$ and $b_1 \not< z$, contradicting the assumption that $z$ is an upper bound of $\downarrow^Z x \cap B$. So, $x \ll_Z b_0$ and therefore $x = \bigvee (\downarrow^Z x \cap B)$. By Definition 3.1, $B$ is a Z-basis for $P$. \qed
Following are some characterizations of $Z$-bases of strongly $Z$-continuous posets.

**Theorem 3.7.** Let $P$ be a $Z$-complete poset and $B \subseteq P$ such that for all $x \in P$, $\downarrow_Z x \cap B \in I_Z(B)$. If the relation $\ll_Z$ has the interpolation property, then the following conditions are equivalent:

1. $B$ is a $Z$-basis for $P$;
2. For any $x, y \in P$ with $x \ll y$, there exists $b \in B$ and $U \in \sigma_Z(P)$ such that $x \in U$, $b \not\ll y$ and $\uparrow b \cup (P \setminus U) = P$;
3. For each $x \in P$, there exists a subset $B_x \subseteq \downarrow_Z x \cap B$ such that $x = \bigvee B_x$.

**Proof.** (1) $\implies$ (2): Let $B$ be a $Z$-basis for $P$. By Proposition 3.3 and the assumption, $P$ is strongly $Z$-continuous. For any $x, y \in P$ with $x \ll y$, it follows from Definition 3.1 that there exists $b \in \downarrow_Z x \cap B$ such that $b \not\ll y$. Thus, it follows from the interpolation property of the relation $\ll_Z$ that there is $t \in P$ such that $b \ll_Z t \ll_Z x$. Let $U = \uparrow_Z t$. It is clear that $x \in U$, $\uparrow b \cup (P \setminus U) = P$ and by Proposition 3.4, we have $U \in \sigma_Z(P)$.

(2) $\implies$ (1): Follows immediately from Definition 3.1 and Lemma 3.6.

(3) $\implies$ (1): Trivial.

(1) $\implies$ (3): Follows immediately from Lemma 3.2. \qed

**Theorem 3.8.** Let $P$ be a strongly $Z$-continuous poset and $B \subseteq P$ such that for all $x \in P$, $\downarrow_Z x \cap B \in I_Z(B)$. Then the following conditions are equivalent:

1. $B$ is a $Z$-basis for $P$;
2. Whenever $x \ll_Z y$, there exists $b \in B$ with $x \ll b \ll_Z y$;
3. Whenever $x \ll_Z y$, there exists $b \in B$ with $x \ll_Z b \ll_Z y$.

**Proof.** (1) $\implies$ (2): Suppose $x \ll_Z y$. It follows from (1) and Lemma 3.2(2) that there exists $S \in Z(B) \subseteq Z(P)$ such that $S \subseteq \downarrow_Z y$ and $y = \bigvee S$. Thus by Definition 2.7, there exists $b \in S \subseteq B$ such that $x \ll b \ll_Z y$.

(2) $\implies$ (3): Suppose $x \ll_Z y$. By (2), there exists $b_0 \in B$ with $x \leq b_0 \ll_Z y$. It follows from the strong $Z$-continuity of $P$ that there is $t \in P$ such that $b_0 \ll_Z t \ll_Z y$. By (2) again, there exists $b \in B$ with $t \leq b \ll_Z y$. It follows from Proposition 2.8(2) that $x \ll_Z b \ll_Z y$.

(3) $\implies$ (1): Follows directly from the $Z$-continuity of $P$ and (3). \qed

4 The $Z$-essential topology

In this section, the essential topology of \textit{dcpo}s in [5] is generalized to the $Z$-essential topology in the setting of $Z$-complete posets. Properties of the $Z$-essential topology and characterizations of $Z$-bases of strongly $Z$-continuous posets via the $Z$-essential topology are obtained.
Definition 4.1. Let $P$ be a poset. Let $\downarrow_Z : 2^P \to 2^P$ be the operator defined by $\downarrow_Z A = \bigcup_{x \in A} \downarrow_Z x$ for all $A \subseteq 2^P$. Let $\uparrow_Z : 2^P \to 2^P$ be the operator defined by $\uparrow_Z A = \bigcup_{x \in A} \uparrow_Z x$ for all $A \subseteq 2^P$.

Proposition 4.2. Let $P$ be a poset. Then for all $A, B \subseteq 2^P$ and $\{A_\alpha\}_{\alpha \in \Gamma} \subseteq 2^P$:

1. $\downarrow_Z \emptyset = \emptyset, \uparrow_Z \emptyset = \emptyset$;
2. $\downarrow_Z (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \downarrow_Z A_\alpha$, $\uparrow_Z (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \uparrow_Z A_\alpha$;
3. $A \subseteq B \implies \downarrow_Z A \subseteq \downarrow_Z B, A \subseteq B \implies \uparrow_Z A \subseteq \uparrow_Z B$;
4. $\downarrow_Z A \setminus \downarrow_Z B \subseteq \downarrow_Z (A \setminus B), \uparrow_Z A \setminus \uparrow_Z B \subseteq \uparrow_Z (A \setminus B)$;
5. $\downarrow_Z (\downarrow_Z A) \subseteq \downarrow_Z A, \uparrow_Z (\uparrow_Z A) \subseteq \uparrow_Z A$.

Proof. Straightforward.

Definition 4.3. Let $P$ be a Z-complete poset and $A \subseteq P$. The subset $A$ is called a Z-essential topology and denoted by $\tau^Z(P)$. Moreover, the intersection of any family of Z-essential topologies is Z-essential.

Proposition 4.4. Let $P$ be a Z-complete poset. Then

1. All the Z-essential sets of $P$ form a topology, called the Z-essential topology and denoted by $\tau^Z_e(P)$. Moreover, the intersection of any family of Z-essential sets is Z-essential;
2. The family of sets $\{\{x\} \cup \downarrow_Z x \mid x \in P\}$ is a base for $\tau^Z_e(P)$;
3. $F \subseteq P$ is Z-essential if and only if $\uparrow_Z F \subseteq F$;
4. For all $A \subseteq 2^P$, $\downarrow_Z A$ is Z-essential, $\uparrow_Z A$ is Z-essential;
5. Every lower set is Z-essential and every upper set is Z-essential;
6. The Z-essential topology $\tau^Z_e(P)$ is finer than $\alpha^*(P)$, i.e., $\alpha^*(P) \subseteq \tau^Z_e(P)$.

Proof. (1) Clearly, $P$ and $\emptyset$ are Z-essential. Let $\{G_\alpha\}_{\alpha \in \Gamma}$ be a family of Z-essential sets. By Proposition 4.2(2), $\downarrow_Z (\bigcup_{\alpha \in \Gamma} G_\alpha) = \bigcup_{\alpha \in \Gamma} \downarrow_Z G_\alpha \subseteq \bigcup_{\alpha \in \Gamma} G_\alpha$. This shows that $\bigcup_{\alpha \in \Gamma} G_\alpha$ is Z-essential. It is easy to see that $\downarrow_Z (\bigcap_{\alpha \in \Gamma} G_\alpha) \subseteq \bigcap_{\alpha \in \Gamma} \downarrow_Z G_\alpha \subseteq \bigcap_{\alpha \in \Gamma} G_\alpha$. So, $\bigcap_{\alpha \in \Gamma} G_\alpha$ is Z-essential.

(2) Straightforward.

(3) Suppose that $F$ is Z-essential. Then $P \setminus F$ is Z-essential and $\downarrow_Z (P \setminus F) \subseteq P \setminus F$. Assume that $\uparrow_Z F \not\subseteq F$. Then there is $x \in \uparrow_Z F$ such that $x \not\in F$. This shows that there exists $y \in F$ such that $y \approx_{Z} x$ and $x \in P \setminus F$. So, $y \in \downarrow_Z (P \setminus F) \subseteq P \setminus F$, a contradiction to $y \in F$. Therefore, we have $\uparrow_Z F \subseteq F$. Conversely, suppose that $\uparrow_Z F \subseteq F$. We only need to show that $P \setminus F$ is Z-essential, i.e., $\downarrow_Z (P \setminus F) \subseteq P \setminus F$. Assume that $\downarrow_Z (P \setminus F) \not\subseteq P \setminus F$. Then there exists $a \in \downarrow_Z (P \setminus F)$ such that $a \not\in P \setminus F$. This shows that there exists $b \in P \setminus F$ such that $a \approx_{Z} b$ and $a \in F$. So, $b \in \uparrow_Z F \subseteq F$, a contradiction to $b \in P \setminus F$. Hence, $\downarrow_Z (P \setminus F) \subseteq P \setminus F$ and $P \setminus F$ is Z-essential.
(4) Follows from (3), Definition 4.3 and Proposition 4.2(5).
(5) Straightforward.
(6) Follows immediately from (5).

**Proposition 4.5.** Let $P$ be a $Z$-complete poset. For all $A \in 2^P$, we have

1. $cl^Z_e(A) = A \cup \uparrow_Z A$, where $cl^Z_e(A)$ is the closure of $A$ in the topology $\tau^Z_e(P)$;
2. $int^Z_e(A) = A \downarrow_Z (P \setminus A)$, where $int^Z_e(A)$ is the interior of $A$ in the topology $\tau^Z_e(P)$;
3. $\uparrow_Z cl^Z_e(A) = cl^Z_e(\uparrow_Z A)$.

**Proof.** (1) By Proposition 4.2, $\uparrow_Z (A \cup \uparrow_Z A) = \uparrow_Z A \cup \uparrow_Z (\uparrow_Z A) = \uparrow_Z A \subseteq (A \cup \uparrow_Z A)$. It follows from Proposition 4.4(3) that the set $A \cup \uparrow_Z A$ is $Z$-e-closed. Let $F$ be any $Z$-e-closed set with $A \subseteq F$. By Proposition 4.2(3) and Proposition 4.4(3), we have $\uparrow_Z A \subseteq \uparrow_Z F \subseteq F$. Therefore, $A \cup \uparrow_Z A \subseteq F$. This shows that $cl^Z_e(A) = A \cup \uparrow_Z A$.

(2) By (1), $cl^Z_e(P \setminus A) = (P \setminus A) \cup \uparrow_Z (P \setminus A)$. Therefore, $int^Z_e(A) = P \setminus cl^Z_e(P \setminus A) = P \setminus ((P \setminus A) \cup \uparrow_Z (P \setminus A)) = A \setminus \uparrow_Z (P \setminus A)$.

(3) Follows immediately from (1) and Proposition 4.4(4).

**Proposition 4.6.** Let $P$ be a strongly $Z$-continuous poset. Then for all $U \in \sigma_Z(P)$ and all $G \in \tau^Z_e(P)$, one has $\uparrow (U \cap G) \in \sigma_Z(P)$.

**Proof.** Let $U \in \sigma_Z(P)$ and $G \in \tau^Z_e(P)$. For all $S \in Z(P)$ with $\sup S \in \uparrow (U \cap G)$, there is $x \in U \cap G$ such that $x \leq \sup S$. By the strong $Z$-continuity of $P$ and Proposition 3.4, there is $t \in U$ such that $t \ll_Z x \leq \sup S$. Thus, there exists $s \in S$ such that $t \leq s$. Since $G \in \tau^Z_e(P)$, we have $t \in \downarrow_Z x \subseteq \downarrow_Z G \subseteq G$. This shows that $t \in U \cap G$ and thus $s \in \uparrow (U \cap G) \cap S$. By Definition 2.4, one has $\uparrow (U \cap G) \in \sigma_Z(P)$.

**Corollary 4.7.** Let $P$ be a strongly $Z$-continuous poset. Then for any $U \in \sigma_Z(P)$ and any lower set $C$, one has $\uparrow (U \cap C) \in \sigma_Z(P)$.

**Lemma 4.8.** Let $P$ be a strongly $Z$-continuous poset. Then the operators $\downarrow_Z$ and $\uparrow_Z$ are idempotent, i.e., for all $A \in 2^P$, one has $\downarrow_Z (\downarrow_Z A) = \downarrow_Z A$, $\uparrow_Z (\uparrow_Z A) = \uparrow_Z A$.

**Proof.** Follows from Proposition 4.2(5) and the strong $Z$-continuity of $P$.

We arrive at giving characterizations of $Z$-bases of a strongly $Z$-continuous poset via the $Z$-essential topology.

**Theorem 4.9.** Let $P$ be a strongly $Z$-continuous poset and $B \subseteq P$ such that for all $x \in P$, $\downarrow_Z x \cap B \in I_Z(B)$. Then the following conditions are equivalent:

1. $B$ is a $Z$-basis of $P$;
(2) \( \uparrow^Z(\uparrow^Z x \cap B) = \uparrow^Z x \) for all \( x \in P \);
(3) \( \uparrow^Z(\uparrow^Z A \cap B) = \uparrow^Z A \) for all \( A \subseteq P \);
(4) \( \uparrow^Z(F \cap B) = \uparrow^Z F \) for all \( Z \)-closed set \( F \);
(5) \( \text{cl}_e^Z(\uparrow^Z A \cap B) = \uparrow^Z A \) for all \( A \subseteq P \);
(6) For all \( U \in \sigma_Z(P) \), \( G \in \tau^Z_e(P) \), \( U \cap G \neq \emptyset \) implies \( U \cap G \cap B \neq \emptyset \).

**Proof.** (1) \( \implies \) (2): By Proposition 4.2, \( \uparrow^Z(\uparrow^Z x \cap B) \subseteq \uparrow^Z(\uparrow^Z x) \subseteq \uparrow^Z x \) for all \( x \in P \). Let \( y \in \uparrow^Z x \). By the strong \( Z \)-continuity of \( P \) and Theorem 3.8, there is \( b \in B \) such that \( x \ll_Z b \ll_Z y \). This shows that \( y \in \uparrow^Z(\uparrow^Z x \cap B) \) and thus \( \uparrow^Z x \subseteq \uparrow^Z(\uparrow^Z x \cap B) \). Therefore, \( \uparrow^Z(\uparrow^Z x \cap B) = \uparrow^Z x \) for all \( x \in P \).

(2) \( \implies \) (3): Let \( A \subseteq P \). By (2) and Proposition 4.2(2), we have \( \uparrow^Z(\uparrow^Z A \cap B) = \uparrow^Z(\bigcup_{z \in A}(\uparrow^Z z \cap B)) = \bigcup_{z \in A} \uparrow^Z(\uparrow^Z z \cap B) = \bigcup_{z \in A} \uparrow^Z z = \uparrow^Z A. \)

(3) \( \implies \) (4): Let \( F \subseteq P \) be a \( Z \)-closed set. By Proposition 4.4(3), we have \( \uparrow^Z F \subseteq F \). Therefore, \( F = \uparrow^Z F \cup (F \setminus \uparrow^Z F) \). By (3) and Proposition 4.2(2), we have

\[
\uparrow^Z(F \cap B) = \uparrow^Z((\uparrow^Z F \cup (F \setminus \uparrow^Z F)) \cap B) = \uparrow^Z((\uparrow^Z F \cap (F \setminus \uparrow^Z F)) \cap B) = \uparrow^Z((F \setminus \uparrow^Z F) \cap B) = \uparrow^Z(F \cap (F \setminus \uparrow^Z F)) = \uparrow^Z F.
\]

(4) \( \implies \) (5): Let \( A \subseteq P \). It follows from (4), Proposition 4.4(4) and Lemma 4.8 that \( \uparrow^Z(\uparrow^Z A \cap B) = \uparrow^Z(\uparrow^Z A) = \uparrow^Z A \). By Proposition 4.5(1), we have that

\[
\text{cl}_e^Z(\uparrow^Z A \cap B) = (\uparrow^Z A \cap B) \cup \uparrow^Z(\uparrow^Z A \cap B) = (\uparrow^Z A \cap B) \cup \uparrow^Z A = \uparrow^Z A.
\]

(5) \( \implies \) (6): For all \( U \in \sigma_Z(P) \), \( G \in \tau^Z_e(P) \) with \( U \cap G \neq \emptyset \), by Proposition 3.4, we have \( U = \uparrow^Z U \). It follows from (5) that \( \text{cl}_e^Z(U \cap B) = \text{cl}_e^Z(\uparrow^Z U \cap B) = \uparrow^Z U = U \). Let \( x \in U \cap G \). Then \( x \in \text{cl}_e^Z(U \cap B) \). Since \( G \in \tau^Z_e(P) \) is a \( Z \)-e-neighborhood of \( x \), we have \( U \cap G \cap B \neq \emptyset \).

(6) \( \implies \) (1): Let \( x \ll_Z y \) in \( P \). By the strong \( Z \)-continuity of \( P \), there exist \( t, z \in P \) such that \( x \ll_Z t \ll_Z z \ll_Z y \). This shows that \( t \in \uparrow^Z x \cap (\{z\} \cup \uparrow^Z z) \). It follows from (6), \( \uparrow^Z x \in \sigma_Z(P) \), \( (\{z\} \cup \uparrow^Z z) \in \tau^Z_e(P) \) and \( \uparrow^Z x \cap (\{z\} \cup \uparrow^Z z) \neq \emptyset \) that \( \uparrow^Z x \cap (\{z\} \cup \uparrow^Z z) \cap B \neq \emptyset \). Pick \( b \in \uparrow^Z x \cap (\{z\} \cup \uparrow^Z z) \cap B \). It is easy to see that \( b \in B \) and \( x \ll_Z b \ll_Z y \). By Theorem 3.8, \( B \) is a \( Z \)-basis of \( P \). \( \square \)

**References**


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