Relative Join Saturation Reductions
of Abstract Knowledge Bases

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Abstract

Abstract knowledge bases (AKB, in short) are generalizations of covering rough sets and knowledge bases. This paper generalizes join saturation reductions to relative join saturation reductions. Relationships between relative join saturation reductions and RU-discernibility matrix / functions are studied. A method to get relative join saturation reductions by RU-discernibility functions for AKB is presented.

Keywords: abstract knowledge base; covering rough sets; relative join saturation reductions; RU-discernibility matrix

1 Introduction and preliminaries

Knowledge reduction is important topic in rough set theory (see [1, 2]). In [3], the transformation and relationship between knowledge reductions of knowledge base and attribute reductions of knowledge expression system were studied. In [4, 5], Xu and Zhao generalized knowledge bases to abstract knowledge bases (AKB, in short) and studied their various reductions. Rong and Xu in [6] defined join saturations and join saturation reductions for AKB, and pointed

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out that as special AKB, knowledge bases can be put into the research framework of AKB naturally. In this paper, the concept of relative join saturation reductions is introduced. Relationships between relative join saturation reductions and RU-discernibility matrix are studied. A method to get relative join saturation reductions by RU-discernibility functions for AKB is presented.

Let $U$ be a nonempty set, $\mathcal{P} \neq \emptyset$ be a family of subsets of $U$. Then $(U, \mathcal{P})$, or $\mathcal{P}$ is called an abstract knowledge base, briefly, an AKB. For an AKB $(U, \mathcal{P})$, we use symbol $\mathcal{P}^\sharp$ to represent the set of all the nonempty finite unions of $\mathcal{P}$. If $\emptyset \neq Q \subseteq \mathcal{P}$, $Q^\sharp = \mathcal{P}^\sharp$, and $\forall P \in Q, (Q - \{P\})^\sharp \neq \mathcal{P}^\sharp$, then $Q$ is called a join saturation reduction of $\mathcal{P}$. In this paper, we use $2^X$ to represent the power set of $X$. Other unstated concepts please refer to [4]-[6].

## 2 Relative join saturation reductions of AKB

In this section, we introduce relative join saturation reductions and related RU-discernibility matrix of an AKB.

**Definition 2.1.** Let $(U, \mathcal{P})$ be an abstract knowledge base, $\emptyset \neq A \subseteq \mathcal{P}^\sharp$. If $\emptyset \neq Q \subseteq \mathcal{P}$ such that $A \subseteq Q^\sharp$, and $\forall P \in Q, A \nsubseteq (Q - \{P\})^\sharp$, then $Q$ is called an $A$-relative join saturation reduction. The set of all the $A$-relative join saturation reductions of $\mathcal{P}$ is denoted by $\text{red}^\sharp_A(\mathcal{P})$.

**Theorem 2.2.** Let $\mathcal{P}$ be an AKB on a finite set $U$, $\emptyset \neq A \subseteq \mathcal{P}^\sharp$. Then $\mathcal{P}$ has at least an $A$-relative join saturation reduction.

**Proof.** By mathematical induction. If $|\mathcal{P}| = 1$, then $\mathcal{P} \in \text{red}^\sharp_A(\mathcal{P})$. Suppose that $\text{red}^\sharp_A(\mathcal{P}) \neq \emptyset$ for all $|\mathcal{P}| \leq k$. Consider the case of $|\mathcal{P}| = k + 1$. If $\mathcal{P}$ satisfies that for all nonempty proper subset $Q$ of $\mathcal{P}$, $A \nsubseteq Q^\sharp$, then $\mathcal{P} \in \text{red}^\sharp_A(\mathcal{P})$. If $\mathcal{P}$ has a nonempty proper subset $W$ such that $A \subseteq W^\sharp$. According to inductive hypothesis, $W$ has an $A$-relative join saturation reduction $Q_0$ such that $A \subseteq Q_0^\sharp$, and that $\forall P \in Q_0, A \nsubseteq (Q_0 - \{P\})^\sharp$. This shows that $Q_0 \in \text{red}^\sharp_A(\mathcal{P})$. According to induction principle, the theorem holds.

**Remark 2.3.** If $A \neq \mathcal{P}^\sharp$, then the $A$-relative join saturation reductions of $\mathcal{P}$ may not be unique. If $A = \mathcal{P}^\sharp$, then the $A$-relative join saturation reductions of $\mathcal{P}$ are the join saturation reductions of $\mathcal{P}$, and by [6, Theorem 3.11], is unique.

For AKB $(U, \mathcal{P})$, $X \subseteq U$ and $\emptyset \neq Q \subseteq \mathcal{P}$, let $\Phi_Q(X) = \{ P \in Q \mid P \subseteq X\}$, then $\Phi_Q : (2^U, \subseteq) \rightarrow (2^Q, \subseteq)$ is order preserving. Clearly, $\Phi_Q(X) = \Phi_P(X) \cap Q$. 
**Proposition 2.4.** Let \((U, \mathcal{P})\) be an AKB, \(\emptyset \neq Q \subseteq \mathcal{P}\).

1. If \(X \in Q^\sharp\), then \(\bigcup \Phi_Q(X) = X\).
2. If \(X, Y \in Q^\sharp\) and \(X \neq Y\), then \(\Phi_Q(X) \neq \Phi_Q(Y)\).

**Proof.** (1) \(\bigcup \Phi_Q(X) \subseteq X\) is obvious. Since \(X \in Q^\sharp\), we have \(\{P_1, P_2, \ldots, P_n\} \subseteq Q\) such that \(X = \bigcup_{i=1}^n P_i\). Noticing that \(P_i \in Q\) and \(P_i \subseteq X\), we have \(P_i \in \Phi_Q(X) = \{P \in Q \mid P \subseteq X\}, \forall i \leq n\). Thus \(X = \bigcup_{i=1}^m P_i \subseteq \bigcup\{P \in Q \mid P \subseteq X\}\), showing that \(X = \bigcup \Phi_Q(X)\).

(2) It is easy to show by (1). \(\square\)

**Proposition 2.5.** Let \((U, \mathcal{P})\) be an AKB, \(\emptyset \neq Q \subseteq \mathcal{P}\).

1. If \(B \subseteq Q\), then \(B \subseteq \Phi_Q(\bigcup B)\).
2. If \(X \subseteq U\), then \(\Phi_Q(\bigcup \Phi_Q(X)) = \Phi_Q(X)\).

**Proof.** If \(B = \emptyset\), then the conclusion is obviously true. Let \(B = \emptyset\). For any \(A \in B\), we have \(A \in Q\) and \(A \subseteq \bigcup B\). So, \(A \in \Phi_Q(\bigcup B) = \{P \in Q \mid P \subseteq \bigcup B\}\).

By the arbitrariness of \(A \in B\), we have \(B \subseteq \Phi_Q(\bigcup B)\).

(2) By (1), we have \(\Phi_Q(X) \subseteq \Phi_Q(\bigcup \Phi_Q(X))\). Because that \(\bigcup \Phi_Q(X) \subseteq X\), so, \(\Phi_Q(\bigcup \Phi_Q(X)) \subseteq \Phi_Q(X)\) and \(\Phi_Q(\bigcup \Phi_Q(X)) = \Phi_Q(X)\). \(\square\)

**Definition 2.6.** Let \(\mathcal{P}\) be an AKB on a finite set \(U\), \(\emptyset \neq A \subseteq \mathcal{P}^\sharp\), \(\mathcal{P}^\sharp \cup \{\perp\} = \{X_1, X_2, \ldots, X_n = \perp\}\), where \(\perp \notin \mathcal{P}^\sharp\) and let \(\perp\) be strictly less than every element of the poset \((\mathcal{P}^\sharp, \subseteq)\). For \(X_i, X_j \in \mathcal{P}^\sharp \cup \{\perp\}\), set

\[
c_{ij} = \begin{cases} 
\Phi_{\mathcal{P}}(X_j) - \Phi_{\mathcal{P}}(X_i), & X_j \in A, X_i \prec X_j, \\
0, & \text{else}
\end{cases}
\]

Then \(D_A(U, \mathcal{P}) = (c_{ij})\) is called \(RU\)-discernibility matrix of \(\mathcal{P}\) relative to \(A\), where \(c_{ij}\) represents the element in row \(i\) and column \(j\) of the matrix \(D_A(U, \mathcal{P})\), \(X_i \prec X_j\) means \(X_i < X_j\) and there is no \(Y \in \mathcal{P}^\sharp\), such that \(X_i < Y < X_j\).

**Lemma 2.7.** Let \(A, B, C\) be three subsets of \(U\). If \(B \subseteq A\), then \((A - B) \cap C \neq \emptyset \iff A \cap C \neq B \cap C\).

**Proof.** It can be verified directly. \(\square\)

**Theorem 2.8.** Let \(\mathcal{P}\) be an AKB on a finite universe \(U\), \(\emptyset \neq A \subseteq \mathcal{P}^\sharp\), \(D_A(U, \mathcal{P}) = (c_{ij})\) is \(RU\)-discernibility matrix of \(\mathcal{P}\) relative to \(A\), and let \(\emptyset \neq Q \subseteq \mathcal{P}\). Then \(A \subseteq Q^\sharp\) if and only if for any \(c_{ij} \neq 0\), one has that \(c_{ij} \cap Q \neq \emptyset\).

**Proof.** \(\Leftarrow:\) For any \(X \in A\), it follows from \(A \subseteq Q^\sharp\) that \(X \in \mathcal{P}^\sharp\). Next we show that \(X \in Q^\sharp\). If \(X \in A\) is a minimal element in \(\mathcal{P}\), then \(\perp \prec X\) and
\[\Phi_P(X) - \Phi_P(\bot) = \{X\}.\] Since \(\emptyset \neq \Phi_P(X) - \Phi_P(\bot) \in D_A(U, \mathcal{P})\), by the assumption, we have \((\Phi_P(X) - \Phi_P(\bot)) \cap Q \neq \emptyset\), that is, \(X \in Q \subseteq Q^\sharp\).

If \(X \in \mathcal{A}\) is not a minimal element in \(\mathcal{P}\), then we can select \(Y\) in \(\mathcal{P}^\sharp\), such that \(Y \preceq X\) and \(\Phi_P(X) - \Phi_P(Y) \in D_A(U, \mathcal{P})\). By Proposition 2.4(2) and order preservation of \(\Phi_P\), we know that \(\Phi_P(Y) \not\subseteq \Phi_P(X)\) and \(\Phi_P(X) - \Phi_P(Y) \neq \emptyset\). By the assumption, \((\Phi_P(X) - \Phi_P(Y)) \cap Q \neq \emptyset\), we have \(\Phi_Q(X) = \Phi_P(X) \cap Q \neq \emptyset\). It follows from \(\emptyset \neq \Phi_Q(X) \subseteq Q\) and \(U\) is finite that \(\bigcup \Phi_Q(X) \in Q^\sharp\). Next we prove \(\bigcup \Phi_Q(X) = X\) by contradiction. Assume that \(\bigcup \Phi_Q(X) \neq X\), then there exists \(Z \in \mathcal{P}^\sharp\) such that \(\bigcup \Phi_Q(X) \subseteq Z \preceq X\). Noticing that \(\Phi_P(X) - \Phi_P(Z) \in D_A(U, \mathcal{P})\) and \(\Phi_P(Z) \not\subseteq \Phi_P(X)\) by Proposition 2.4(2), we have \(\Phi_P(X) - \Phi_P(Z) \neq \emptyset\). By the assumption, \((\Phi_P(X) - \Phi_P(Z)) \cap Q \neq \emptyset\). It follows from \(\bigcup \Phi_Q(X) \subseteq Z\) that \(\Phi_P(\bigcup \Phi_Q(X)) \subseteq \Phi_P(Z)\). Thus \((\Phi_P(X) - \Phi_P(\bigcup \Phi_Q(X))) \cap Q \neq \emptyset\). By Lemma 2.7 and \(\Phi_P(\bigcup \Phi_Q(X)) \subseteq \Phi_P(X)\), we know that \(\Phi_P(X) \cap Q \neq \Phi_P(\bigcup \Phi_Q(X)) \cap Q\), namely \(\Phi_Q(X) \neq \Phi_Q(\bigcup \Phi_Q(X))\), contradicting Proposition 2.5(2). Thus \(\bigcup \Phi_Q(X) = X\). Since \(\bigcup \Phi_Q(X) \in Q^\sharp\), we have \(\mathcal{A} \subseteq Q^\sharp\) by the arbitrariness of \(X \in \mathcal{A}\).

\[\Rightarrow:\] By Definition 2.6, for any \(c_{ij} \in D_A(U, \mathcal{P})\) with \(c_{ij} \neq \emptyset\), there exists \(X_i, X_j \in \mathcal{P}^\sharp \cup \{\bot\}\) such that \(c_{ij} = \Phi_P(X_j) - \Phi_P(X_i)\), where \(X_j \in \mathcal{A} \subseteq Q^\sharp\) and \(X_i \preceq X_j\). If \(X_j\) is a minimal element in \(\mathcal{P}\), then \(X_i = \bot\), \(\Phi_P(X_i) = \emptyset\) and \(\Phi_P(X_j) = \{X_j\}\). Because \(X_j \in Q^\sharp\) is a minimal element in \(\mathcal{P}\), so \(X_j \in Q\). Thus \(\Phi_P(X_j) \cap Q = \{X_j\}\). And then \(Q \cap (\Phi_P(X_j) - \Phi_P(X_i)) = Q \cap \Phi_P(X_j) \neq \emptyset\), namely, \(c_{ij} \cap Q \neq \emptyset\). If \(X_j\) is not a minimal element in \(\mathcal{P}\), then \(X_j \neq \bot\) and \(X_j \in \mathcal{P}^\sharp\). It follows from Proposition 2.4(1) and \(X_j \in Q^\sharp\) that \(\bigcup \Phi_Q(X_j) = X_j\). By \(\bigcup \Phi_Q(X_i) \subseteq X_i \preceq X_j\), we know that \(\bigcup \Phi_Q(X_i) \subseteq X_i \subseteq X_j = \bigcup \Phi_Q(X_j)\). Noticing that \(\bigcup \Phi_Q(X_i) \not\subseteq \bigcup \Phi_Q(X_j)\), we have \(\Phi_Q(X_j) \neq \Phi_Q(X_i)\). And by Lemma 2.7, we have \(Q \cap (\Phi_P(X_j) - \Phi_P(X_i)) \neq \emptyset\), namely, \(c_{ij} \cap Q \neq \emptyset\). \(\square\)

The following is the criterion theorem of relative join saturation reductions.

**Theorem 2.9.** Let \(\mathcal{P}\) be an AKB on a finite set \(U\), \(\emptyset \neq \mathcal{A} \subseteq \mathcal{P}^\sharp\), \(D_A(U, \mathcal{P}) = (c_{ij})\) is the RU-discernibility matrix of \(\mathcal{P}\) relative to \(\mathcal{A}\). Then \(\mathcal{Q} \in \text{red}^\sharp_A(\mathcal{P})\) iff \(\mathcal{Q}\) is a minimal subfamily of \(\mathcal{P}\) satisfying for any \(\emptyset \neq c_{ij} \in D_A(U, \mathcal{P})\), \(\mathcal{Q} \cap c_{ij} \neq \emptyset\).

**Proof.** It follows directly by Definition 2.1 and Theorem 2.8. \(\square\)

### 3 Method to get relative join saturation reductions

In this section, we introduce RU-discernibility functions of AKB on a finite set. A method to get join saturation reductions is presented.
Definition 3.1. Let $\mathcal{P}$ be an AKB on a finite set $U$, $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}^2$, $D_A(U, \mathcal{P}) = (c_{ij})$ is RU-discernibility matrix of $\mathcal{P}$ relative to $\mathcal{A}$, where $1 \leq i, j \leq n$. Denote the Boolean function $\bigwedge_{c_{ij}, \neq \emptyset} \left( \bigvee_{P \in c_{ij}} P \right)$ by $f_A(\mathcal{P})$, then we call $f_A(\mathcal{P})$ the RU-discernibility function of $\mathcal{P}$ relative to $\mathcal{A}$.

Definition 3.2. Let $\mathcal{P}$ be an AKB on a finite set $U$, $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}^2$, $\bigvee_{k=1}^{m} \left( \bigwedge_{P \in \mathcal{P}_k} P \right)$ is a disjunctive normal form of $f_A(\mathcal{P})$. If $\bigcup_{k=1}^{m} \mathcal{P}_k \subseteq \mathcal{P}$, and $\forall k, j \leq m$ when $k \neq j$, $\mathcal{P}_k$ and $\mathcal{P}_j$ do not contain each other. Then we call this disjunctive normal form is a minimal disjunctive normal form of $f_A(\mathcal{P})$.

Example 3.3. Let $f_A(\mathcal{P}) = P_3 \lor (P_1 \lor P_2) \lor (P_1 \lor P_3) \lor (P_2 \lor P_3) \lor (P_1 \lor P_2 \lor P_3)$. By Boolean logic operation law, we know that $f_A(\mathcal{P}) = P_3 \lor (P_1 \lor P_2) = (P_1 \land P_3) \lor (P_2 \land P_3)$. By Definition 3.2 we know that $(P_1 \land P_3) \lor (P_2 \land P_3)$ is a minimal disjunctive normal form of $f_A(\mathcal{P})$.

Theorem 3.4. Let $\mathcal{P}$ be an AKB on a finite set $U$, $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}^2$, $\bigvee_{k=1}^{m} \left( \bigwedge_{P \in \mathcal{P}_k} P \right)$ is a minimal disjunctive normal form of $f_A(\mathcal{P})$, $\emptyset \neq \mathcal{Q} \subseteq \mathcal{P}$. Then $\mathcal{Q} \in \text{red}^2_A(\mathcal{P})$ if and only if there exists $k \leq m$ such that $\mathcal{Q} = \mathcal{P}_k$.

Proof. It can be proved directly by using Boolean logic operation law.

Remark 3.5. Theorem 3.4 gives a method to get relative join saturation reductions: using Boolean logic operation law to transform $f_A(\mathcal{P})$ into minimal disjunctive normal form $\bigvee_{k=1}^{m} \left( \bigwedge_{P \in \mathcal{P}_k} P \right)$, then $\{\mathcal{P}_k \mid k = 1, 2, \ldots, m\}$ are precisely $\mathcal{A}$-relative join saturation reductions of AKB $\mathcal{P}$.

The following is a concrete example to illustrate the method to get relative join saturation reductions of an AKB.

Example 3.6. Let $U = \{x_1, x_2, \ldots, x_6\}$, $\mathcal{P} = \{P_1, P_2, \ldots, P_6\}$, where $P_1 = \{x_1, x_2\}$, $P_2 = \{x_2, x_3\}$, $P_3 = \{x_1, x_3\}$, $P_4 = \{x_1, x_2, x_3\}$, $P_5 = \{x_2, x_3, x_4\}$, $P_6 = \{x_5, x_6\}$. Then it is easy to check that $\mathcal{P}^2 = \{X_1, X_2, \ldots, X_{13}\}$, where $X_1 = \{x_1, x_2\}$, $X_2 = \{x_2, x_3\}$, $X_3 = \{x_1, x_3\}$, $X_4 = \{x_1, x_2, x_3\}$, $X_5 = \{x_2, x_3, x_4\}$, $X_6 = \{x_5, x_6\}$, $X_7 = \{x_1, x_2, x_3, x_4\}$, $X_8 = \{x_1, x_2, x_5, x_6\}$, $X_9 = \{x_2, x_3, x_5, x_6\}$, $X_{10} = \{x_1, x_3, x_5, x_6\}$, $X_{11} = \{x_1, x_2, x_3, x_5, x_6\}$, $X_{12} = \{x_2, x_3, x_4, x_5, x_6\}$, $X_{13} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Set $X_{14} = \bot$. Let $\mathcal{A} = \{X_1, X_4, X_6, X_7, X_{11}\}$. By Definition 2.6, if $X_k \in \mathcal{P}^2 \cup \{\bot\} - \{\mathcal{A}\}$, then the elements in the $k^{th}$ column are empty sets. So we only list the columns corresponding to the elements in $\mathcal{A}$. 
The RU-discernibility matrix $D_A(U, \mathcal{P})$ of $\mathcal{P}$

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & X_1 & X_4 & X_6 & X_7 \\
\hline
X_1 & \emptyset & \{P_2, P_3, P_4\} & \emptyset & \emptyset \\
X_2 & \emptyset & \{P_1, P_3, P_4\} & \emptyset & \emptyset \\
X_3 & \emptyset & \{P_1, P_2, P_4\} & \emptyset & \emptyset \\
X_4 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_5 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_6 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_7 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_8 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_9 & \emptyset & \emptyset & \emptyset & \emptyset \\
X_{10} & \emptyset & \emptyset & \emptyset & \emptyset \\
X_{11} & \emptyset & \emptyset & \emptyset & \emptyset \\
X_{12} & \emptyset & \emptyset & \emptyset & \emptyset \\
X_{13} & \emptyset & \emptyset & \emptyset & \emptyset \\
\hline
\end{array}
\]

$\bot$ \{\{P_1\}\} \emptyset \{P_6\} \emptyset \emptyset

Transform $f_A(\mathcal{P})$ to minimal disjunctive normal form as follows

\[
f_A(\mathcal{P}) = P_1 \land P_6 \land (P_2 \lor P_3 \lor P_4) \land (P_1 \lor P_3 \lor P_4) \land (P_1 \lor P_2 \lor P_4) \land P_5
\]

\[
= (P_1 \land P_6 \land P_5) \land (P_2 \lor P_3 \lor P_4)
\]

\[
= (P_1 \land P_2 \land P_5 \land P_6) \lor (P_1 \land P_3 \land P_5 \land P_6) \lor (P_1 \land P_4 \land P_5 \land P_6).
\]

We have $\text{red}_A^\sharp(\mathcal{P}) = \{\{P_1, P_2, P_5, P_6\}, \{P_1, P_3, P_5, P_6\}, \{P_1, P_4, P_5, P_6\}\}$.

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