Exploring the Vertex and Edge Corona of Graphs for their Weakly Connected 2-Domination

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Abstract

A weakly connected 2-dominating set of a connected graph $G$ is a set $D \subseteq V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to at least two vertices in $D$ and the subgraph $(D)_w$, which is the one weakly induced by $D$, is connected. In this paper, the weakly connected 2-dominating sets in the vertex corona $G \circ H$ and edge corona $G \diamond H$ of two graphs $G$ and $H$ are characterized. As a consequence, it is shown that if $G$ and $H$ are any nontrivial graphs with $G$ connected of order $m$, then $\gamma_2^w(G \circ H) = \min \{ \gamma_w(G) + m\gamma_2(H), m(1 + \gamma(H)) \}$ and $\gamma_2^w(G \diamond H) = m$, where $\gamma$, $\gamma_2$, $\gamma_w$ and $\gamma_2^w$ are the domination number, 2-domination number, weakly connected domination number and weakly connected 2-domination number of the concerned graphs, respectively.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. The set of neighbors of a vertex $u \in V(G)$ is called the open neighborhood of $u$ and is denoted by $N_G(u)$ and the closed neighborhood of $u$ is the set $N_G[u] = N_G(u) \cup \{u\}$. The open neighborhood of $U \subseteq V(G)$ is the set $N_G(U) = \bigcup_{u \in U} N_G(u)$ and the closed neighborhood of $U$ is $N_G[U] = U \cup N_G(U)$. A subset $D$ of $V(G)$ is called weakly connected if the subgraph $\langle D \rangle_w = (N_G[D], E_w)$ weakly induced by $D$ is connected, where $E_w$ consists of all edges in $G$ incident with at least one vertex in $D$.

A set $D \subseteq V(G)$ is a dominating set in $G$ if for every $u \in V(G) \setminus D$, there exists $v \in D$ such that $uv \in E(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set in $G$. A dominating set in $G$ with cardinality $\gamma(G)$ is called a $\gamma$-set of $G$. Moreover, a dominating set $D \subseteq V(G)$ is a weakly connected dominating set in $G$ if the subgraph $\langle D \rangle_w$ weakly induced by $D$ is connected. The weakly connected domination number of $G$, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set in $G$. A weakly connected dominating set in $G$ with cardinality $\gamma_w(G)$ is called a $\gamma_w$-set of $G$.

A set $D \subseteq V(G)$ is a 2-dominating set in $G$ if for every $u \in V(G) \setminus D$, $|D \cap N_G(u)| \geq 2$. The 2-domination number of $G$, denoted by $\gamma_2(G)$, is the smallest cardinality of a 2-dominating set in $G$. A 2-dominating set in $G$ with cardinality $\gamma_2(G)$ is called a $\gamma_2$-set of $G$. A 2-dominating set $D \subseteq V(G)$ is a weakly connected 2-dominating set in $G$ if the subgraph $\langle D \rangle_w$ weakly induced by $D$ is connected. The minimum cardinality of a weakly connected 2-dominating set in $G$ is the weakly connected 2-domination number $\gamma_{2w}(G)$. Any weakly connected 2-dominating set in $G$ whose cardinality is equal to $\gamma_{2w}(G)$ is called a $\gamma_{2w}$-set of $G$.

The concept of weakly connected 2-domination was introduced in [5]. Militante et al. in [7] explored the concept of weakly connected 2-domination in the join $G + H$ of graphs $G$ and $H$, where some relevant bounds for the weakly connected 2-domination number of $G + H$ were obtained.

For the corona or vertex corona of graphs, this binary operation was first introduced by Frucht and Harary in [3] as cited in [1]. The edge corona, on the other hand, was introduced by Hou and Shiu in [4] where the spectrum and the number of spanning trees in the edge corona were discussed.

In this paper, the weakly connected 2-domination in the corona of graphs and in the edge corona of graphs are investigated. In particular, the weakly connected 2-dominating sets in the corona $G \circ H$ and edge corona $G \diamond H$ are characterized and their corresponding weakly connected 2-domination numbers are obtained. Throughout this paper, every graph is understood within the context of being simple, finite and undirected. Other graph terminologies which are not specifically defined in this paper can be found in [2].
2 Results in the Vertex Corona of Graphs

We shall need the following known results in this study.

**Theorem 2.1** [7] Let \( K_1 = \langle \{v\} \rangle \) and \( H \) be any graph of order at least 2. Then \( D \subseteq V(K_1 + H) \) is a weakly connected 2-dominating set of \( K_1 + H \) if and only if one of the following holds:

(i) \( v \in D \) and \( D \setminus \{v\} \) is a dominating set of \( H \).

(ii) \( D \subseteq V(H) \) and \( D \) is a 2-dominating set of \( H \).

**Remark 2.2** [7] Let \( G \) and \( H \) be any graphs. If \( D \) is a nonempty subset of \( V(G + H) \), then \( \langle D \rangle_w \) is connected.

Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively. The *corona* \( G \circ H \) of \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \), and then joining the \( i \)th vertex of \( G \) to every vertex of the \( i \)th copy of \( H \). For every \( v \in V(G) \), denote by \( H^v \) the copy of \( H \) whose vertices are attached one by one to the vertex \( v \). Denote by \( v + H^v \) the subgraph of the corona \( G \circ H \) corresponding to the join \( \langle \{v\} \rangle + H^v \). As an illustration, the corona of the path \( P_3 \) and the complete graph \( K_4 \) is given in Figure 1 below.

![Figure 1](image-url)

Figure 1: The path \( P_3 \), the complete graph \( K_4 \) and the corona \( P_3 \circ K_4 \), with some darkened vertices constituting a particular \( \gamma_{2w} \)-set in \( P_3 \circ K_4 \).

The following result characterizes the weakly connected 2-dominating sets in the corona \( G \circ H \).

**Theorem 2.3** Let \( G \) and \( H \) be nontrivial graphs with \( G \) connected. Then \( C \subseteq V(G \circ H) \) is a weakly connected 2-dominating set in \( G \circ H \) if and only if \( C = D \cup \bigcup_{v \in D} S^v \) \( \cup \bigcup_{a \in V(G) \setminus D} T^a \), where \( D \) is a weakly connected dominating set in \( G \), \( S^v \) is a dominating set in \( H^v \) for every \( v \in D \), and \( T^a \) is a 2-dominating set in \( H^a \) for every \( a \in V(G) \setminus D \). In particular, if \( D = V(G) \), then \( C = V(G) \cup \bigcup_{v \in V(G)} S^v \), where \( S^v \) is a dominating set in \( H^v \) for every \( v \in V(G) \).

**Proof:** We note that \( G \circ H \) is connected since \( G \) is connected. Suppose that \( C \subseteq V(G \circ H) \) is a weakly connected 2-dominating set in \( G \circ H \). Then clearly \( C \) is a weakly connected dominating set in \( G \circ H \). Observe that \( C \cap V(G) \) is not
empty because if \( C \cap V(G) = \emptyset \), then \( \langle C \rangle_w \) would be disconnected, contrary to our assumption. If \( C \cap V(G) \) were not a dominating set in \( G \), then there would exist \( v \in V(G) \setminus C \) such that \( vw \notin E(G) \) for every \( u \in C \cap V(G) \). This means that \( \langle C \cap V(v + H^v) \rangle_w \) would be in a separate component of \( \langle C \rangle_w \) so that \( \langle C \rangle_w \) would be disconnected, which is again contrary to our assumption. Hence, \( C \cap V(G) \) is a dominating set in \( G \). If \( \langle C \cap V(G) \rangle_w \) were not connected in \( G \), then by definition of the corona \( G \circ H \), \( \langle C \rangle_w \) would have been disconnected in \( G \circ H \). This is again a contradiction to the assumption. So, it follows that \( \langle C \cap V(G) \rangle_w \) is connected in \( G \). Thus, combining the aforementioned observations, \( C \cap V(G) \) must now be a weakly connected dominating set in \( G \).

Set \( D = C \cap V(G) \).

Let \( v \in D \). Since \( v + H^v \) is isomorphic to \( K_1 + H \) and \( C \) is a weakly connected 2-dominating set in \( G \circ H \), \( C \cap V(v + H^v) \) is a weakly connected 2-dominating set in \( v + H^v \). By Theorem 2.1 (i), \( C \cap V(H^v) \) is a dominating set in \( H^v \) for every \( v \in D \). Set \( S^a = C \cap V(H^v) \) for every \( v \in D \).

Let \( a \in V(G) \setminus D \). Since \( a + H^a \cong K_1 + H \) and \( C \) is a weakly connected 2-dominating set in \( G \circ H \), \( C \cap V(a + H^a) \) is a weakly connected 2-dominating set in \( a + H^a \). By Theorem 2.1 (ii), \( C \cap V(a + H^a) \) is a 2-dominating set in \( H^a \) for each \( a \in V(G) \setminus D \). Set \( T^a = C \cap V(H^a) \) for every \( a \in V(G) \setminus D \).

Combining the three properties above, we obtain \( C = D \cup (\bigcup_{v \in D} S^v) \cup (\bigcup_{a \in V(G) \setminus D} T^a) \), where \( D \) is a weakly connected dominating set in \( G \), \( S^v \) is a dominating set in \( H^v \) for every \( v \in D \), and \( T^a \) is a 2-dominating set in \( H^a \) for every \( a \in V(G) \setminus D \).

For the converse, suppose that \( C = D \cup (\bigcup_{v \in D} S^v) \cup (\bigcup_{a \in V(G) \setminus D} T^a) \) where \( D, S^v \) and \( T^v \) satisfy the given properties. We show first that \( \langle C \rangle_w \) is connected. Since \( C \cap V(v + H^v) \subseteq V(v + H^v) \), by Remark 2.2, \( \langle C \cap V(v + H^v) \rangle_w \) is connected in \( v + H^v \) for every \( v \in D \). Similarly, \( \langle C \cap V(a + H^a) \rangle_w \) is connected in \( a + H^a \) for every \( a \in V(G) \setminus D \). Thus we have the following: \( \langle C \cap V(v + H^v) \rangle_w \) is connected in \( v + H^v \) for every \( v \in D \), \( \langle C \cap V(a + H^a) \rangle_w \) is connected in \( a + H^a \) for every \( a \in V(G) \setminus D \) and \( \langle D \rangle_w \) is connected in \( G \circ H \).

Since \( \langle C \rangle_w = \left( D \cup (\bigcup_{v \in D} S^v) \cup (\bigcup_{a \in V(G) \setminus D} T^a) \right)_w \), \( \langle D \rangle_w \) is connected and \( V(G) \subseteq V(\langle D \rangle_w) \), necessarily \( \langle C \rangle_w \) must be connected in \( G \circ H \).

Now to show that \( C \) is a 2-dominating set in \( G \circ H \), note that the vertices \( b \) in \( V(G \circ H) \setminus C \) come from three possible sources, namely, \( b \in V(G) \setminus D \), \( b \in V(H^a) \) where \( a \in V(G) \setminus D \) and \( b \in V(H^v) \) where \( v \in D \).

Case 1. \( b \in V(G) \setminus D \). Since \( T^b \) is a 2-dominating set in \( H^b \) and \( |V(H)| \geq 2 \), we have \( |T^b| \geq 2 \). Also, \( T^b \subseteq N_{G \circ H}(b) \). Thus, \( b \) is adjacent to at least two vertices in \( C \) since \( T^b \subseteq C \).

Case 2. \( b \in V(H^a) \) where \( a \in V(G) \setminus D \). Since \( T^a \) is a 2-dominating set in \( H^a \), we have \( T^a \subseteq N_{a + H^v}(b) \). It means that \( b \) is dominated by at least two vertices in \( T^a \subseteq C \).
Case 3. $b \in V(H^v)$ where $v \in D$. Since $S^v$ is a dominating set in $H^v$ for every $v \in D$, there exists $u \in S^v \subseteq C$ such that $bu \in E(H^v)$. By definition of the corona of two graphs, $bv \in E(v + H^v)$. Hence, $u, v \in N_{G \circ H}(b)$. Of course, $u, v \in C$. This means that $b$ is dominated by at least two vertices in $C$.

From the three cases above, one can see that every vertex in $V(G \circ H) \setminus C$ is dominated by at least two vertices in $C$. That is, $C$ is a 2-dominating set in $G \circ H$. Hence, $C$ is now a weakly connected 2-dominating set in $G \circ H$. In particular, if $D = V(G)$, then $\bigcup_{a \in V(G) \setminus D} T^a = \emptyset$. This particular case means that $C = D \cup \left( \bigcup_{v \in V(G)} S^v \right)$ where $S^v$ is a dominating set in $H^v$ for every $v \in V(G)$. □

The following result is a consequence of Theorem 2.3.

**Corollary 2.4** Let $G$ and $H$ be nontrivial graphs with $G$ connected of order $m \geq 2$. Then the weakly connected 2-domination number of $G \circ H$ is given by

$$\gamma_{2w}(G \circ H) = \begin{cases} \gamma_w(G) + m\gamma_2(H), & \text{if } \gamma_2(H) = \gamma(H) \\ m(1+\gamma(H)), & \text{if } \gamma_2(H) > \gamma(H). \end{cases}$$

That is, \( \gamma_{2w}(G \circ H) = \min \{ \gamma_w(G) + m\gamma_2(H), m(1+\gamma(H)) \} \).

**Proof:** Suppose that $C$ is a $\gamma_{2w}$-set in $G \circ H$. Then by Theorem 2.3, $C = D \cup \left( \bigcup_{v \in D} S^v \right) \cup \left( \bigcup_{a \in V(G) \setminus D} T^a \right)$, where $D$ is a weakly connected dominating set in $G$, $S^v$ is a $\gamma$-set in $H^v$ for every $v \in D$, and $T^a$ is a $\gamma_2$-set in $H^a$ for every $a \in V(G) \setminus D$. Using the fact that $\gamma(H) \leq \gamma_2(H)$, we consider three cases:

Case 1. $\gamma_2(H) = \gamma(H)$. Since $C$ is a $\gamma_{2w}$-set in $G \circ H$, it is necessary that the set $D$ in the expression for $C$ above must be a $\gamma_w$-set in $G$ so that $\gamma_{2w}(G \circ H) = |C| = \gamma_w(G) + m\gamma_2(H)$.

Case 2. $\gamma_2(H) = \gamma(H) + 1$. In this case, the set $D$ in the expression for $C$ can just be any weakly connected dominating set in $G$ so that

$$\gamma_{2w}(G \circ H) = |C| = |D| + |D|\gamma(H) + (m - |D|)\gamma_2(H)$$

$$\quad = |D|(1 + \gamma(H)) + m(1 + \gamma(H)) - |D|(1 + \gamma_2(H))$$

$$\quad = m(1 + \gamma(H)).$$

Case 3. $\gamma_2(H) - \gamma(H) \geq 2$. In this case, the only possibility for the set $D$ in the expression for $C$ above is $D = V(G)$ so that $\gamma_{2w}(G \circ H) = |C| = m + m\gamma(H) = m(1 + \gamma(H))$.

From all the three cases above, we can see that

$$\gamma_{2w}(G \circ H) = \begin{cases} \gamma_w(G) + m\gamma_2(H), & \text{if } \gamma_2(H) = \gamma(H) \\ m(1+\gamma(H)), & \text{if } \gamma_2(H) > \gamma(H). \end{cases}$$
That is, $$\gamma_{2w}(G \circ H) = \min \{\gamma_w(G) + m\gamma_2(H), m(1 + \gamma(H))\}.$$ \hfill \Box

The next results are immediate consequences of Corollary 2.4.

**Remark 2.5** Let $$G$$ be a connected graph of order $$m \geq 2$$ and let $$K_n$$ be the complete graph of order $$n \geq 2$$. Then $$\gamma_{2w}(G \circ K_n) = 2m$$.

**Remark 2.6** Let $$G$$ be a connected graph of order $$m \geq 2$$ and let $$\overline{K}_n$$ be the empty graph of order $$n \geq 2$$. Then $$\gamma_{2w}(G \circ \overline{K}_n) = \gamma_w(G) + mn$$.

The next three remarks also follow directly from Corollary 2.4, the first two of which coincide exactly with some of the results given in [6]. Since these remarks include some special families of graphs such as the centipede, bi-star, and barbell graphs, it is just proper that we recall their basic shapes or structures as suggested in the samples given in Figure 2 below.

![Figure 2](image-url)

Figure 2: (a) centipede $$c_{4,2}$$; (b) bi-star $$B(5,5)$$; and the (c) barbell graph $$B_{4,4}$$.

**Remark 2.7** Let $$c_{n,2}$$ be a centipede graph with $$n \geq 2$$. Then the weakly connected 2-domination number of $$c_{n,2}$$ is given by $$\gamma_{2w}(c_{n,2}) = \left\lfloor \frac{n}{4} \right\rfloor + 2n$$.

**Proof**: Since $$c_{n,2} \cong P_n \circ \overline{K}_2$$, we have $$\gamma_{2w}(c_{n,2}) = \gamma_{2w}(P_n \circ \overline{K}_2)$$. Since $$\gamma_w(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$$ and $$\gamma_2(\overline{K}_2) = 2$$, by Corollary 2.4, we have $$\gamma_{2w}(c_{n,2}) = \gamma_w(P_n) + |V(P_n)|\gamma_2(\overline{K}_2) = \left\lfloor \frac{n}{2} \right\rfloor + 2n$$.

**Remark 2.8** Let $$B(n,n)$$ be a bi-star graph with $$n \geq 2$$. Then the weakly connected 2-domination number of $$B(n,n)$$ is given by $$\gamma_{2w}(B(n,n)) = 2n + 1$$.

**Proof**: Since $$B(n,n) \cong P_2 \circ \overline{K}_n$$, we have $$\gamma_{2w}(B(n,n)) = \gamma_{2w}(P_2 \circ \overline{K}_n)$$. Since $$\gamma_w(P_2) = 1$$ and $$\gamma(\overline{K}_n) = \gamma_2(\overline{K}_n) = n$$, by Corollary 2.4, we have $$\gamma_{2w}(B(n,n)) = \gamma_w(P_2) + |V(P_2)|\gamma_2(\overline{K}_n) = 1 + 2n$$.

**Remark 2.9** Let $$B_{n,n}$$ be a barbell graph with $$n \geq 3$$. Then $$\gamma_{2w}(B_{n,n}) = 4$$.

**Proof**: Since $$B_{n,n} \cong P_2 \circ K_{n-1}$$, we have $$\gamma_{2w}(B_{n,n}) = \gamma_{2w}(P_2 \circ K_{n-1})$$. Also, since $$\gamma_w(P_2) = 1$$, $$\gamma(K_{n-1}) = 1$$ and $$\gamma_2(K_{n-1}) = 2$$, by Corollary 2.4, we have $$\gamma_{2w}(P_2 \circ K_{n-1}) = |V(P_2)|\left(1 + \gamma(K_{n-1})\right) = 2(1 + 1) = 4$$. \hfill \Box
3 Results in the Edge Corona of Graphs

Let $G$ and $H$ be two graphs on disjoint sets of $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges, respectively. The edge corona $G \circ H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $m_1$ copies of $H$ and then joining the two end-vertices of the $i$th edge of $G$ to every vertex in the $i$th copy of $H$. Note that the edge corona $G \circ H$ of $G$ and $H$ has $n_1 + m_1 n_2$ vertices and $m_1 + 2m_1n_2 + m_1m_2$ edges [4]. If $ab \in E(G)$, then the copy $H$ whose vertices are connected one by one to both $a$ and $b$ in $G \circ H$ is called the $ab$-copy of $H$ and is denoted by $H_{ab}$. Moreover, if $V(H) = \{v_1, v_2, ..., v_n\}$, then the vertices of $H_{ab}$ may be denoted by $v_{1a}^{ab}, v_{2a}^{ab}, ..., v_{na}^{ab}$. Notice that $\langle \{a, b\} \cup V(H_{ab}) \rangle_{G \circ H} \cong P_2 + H$. As an illustration, let $G$ be the path $P_3$ and $H$ be the cycle $C_3$. The two edge coronas $P_3 \circ C_3$ and $C_3 \circ P_3$ are drawn in Figure 3.

![Graphs](image)

Figure 3: The path $P_3$, the cycle $C_3$ and the edge coronas $C_3 \circ P_3$ and $P_3 \circ C_3$, with darkened vertices in some $\gamma_{2w}$-sets.

**Remark 3.1** If $a_1b_1$ and $a_2b_2$ are two distinct edges of $G$, then it is clear that $V(H_{a_1b_1}) \cap V(H_{a_2b_2}) = \emptyset$.

**Theorem 3.2** Let $G$ and $H$ be nontrivial graphs of orders $m$ and $n$, respectively, with $G$ connected. Then the set $V(G)$ is a weakly connected 2-dominating set in $G \circ H$. As a consequence, $\gamma_{2w}(G \circ H) \leq m$.

**Proof**: From the definition of the edge corona $G \circ H$, it is clear that $V(G)$ is a 2-dominating set in $G \circ H$ (see for example Figure 3). Moreover, the subgraph $V(G)_{w}$ of $G \circ H$ is exactly what remains from $G \circ H$ after removing all the edges between any two adjacent vertices in the copies of $H$. Since $G$ is connected by assumption, $V(G)_{w}$ is also connected. Thus, $V(G)$ is a weakly connected 2-dominating set in $G \circ H$. As a consequence, $\gamma_{2w}(G \circ H) \leq m$. □

**Theorem 3.3** Let $G$ and $H$ be nontrivial graphs with $G$ connected. Then
$T \subseteq V(G \odot H)$ is a weakly connected 2-dominating set in $G \odot H$ if and only if

$$
T = S \cup \left( \bigcup_{a,b \in V(G) \setminus S \atop ab \in E(G)} T_{0}^{[ab]} \right) \cup \left( \bigcup_{a,b \in V(G), ab \in E(G) \atop |\{a,b\}\cap S|=1} T_{1}^{[ab]} \right) \cup \left( \bigcup_{a,b \in S \atop ab \in E(G)} T_{2}^{[ab]} \right),
$$

where $S \subseteq V(G)$, $T_{0}^{[ab]}$ is a 2-dominating set in $H^{ab}$ for every pair $a$ and $b$ in $V(G) \setminus S$ with $ab \in E(G)$, $T_{1}^{[ab]}$ is a dominating set in $H^{ab}$ for every pair $a$ and $b$ in $V(G)$ with $|\{a,b\}\cap S|=1$ and $ab \in E(G)$, and $T_{2}^{[ab]} \subseteq V(H^{ab})$ for every pair $a$ and $b$ in $S$ with $ab \in E(G)$.

Proof: Suppose $T \subseteq V(G \odot H)$ is a weakly connected 2-dominating set in $G \odot H$. Let $S = T \cap V(G)$ and let $a, b \in V(G)$ such that $ab \in E(G)$. Since $V(H^{a,b_{1}}) \cap V(H^{a,b_{2}}) = \emptyset$ for every two distinct edges $a_{1}b_{1}, a_{2}b_{2} \in E(G)$ by Remark 3.1, it follows that $T \cap V(\langle \{a, b\} \cup V(H^{ab}) \rangle)$ is nonempty. Moreover, due to the adjacency of the vertices in the subgraph $\langle \{a, b\} \cup V(H^{ab}) \rangle$ (see for instance Figure 3), it can be seen that $T \cap V(\langle \{a, b\} \cup V(H^{ab}) \rangle)$ is a weakly connected 2-dominating set in $\langle \{a, b\} \cup V(H^{ab}) \rangle$. Let us consider the cases where $|\{a, b\}\cap S|=0, 1, 2$.

Case 1. Suppose $a, b \in V(G) \setminus S$. Then $T \cap V(H^{ab})$ must be a 2-dominating set in $H^{ab}$. Set $T_{0}^{[ab]} = T \cap V(H^{ab})$ for every pair $a, b \in V(G) \setminus S$ with $ab \in E(G)$.

Case 2. Suppose $|\{a, b\}\cap S|=1$. Then necessarily $T \cap V(H^{ab})$ is a dominating set in $H^{ab}$. Set $T_{1}^{[ab]} = T \cap V(H^{ab})$ for every pair $a, b \in V(G)$ with $ab \in E(G)$ and $|\{a, b\}\cap S|=1$.

Case 3. Let $a, b \in S$. Then clearly $T \cap V(H^{ab}) \subseteq V(H^{ab})$. Set $T_{2}^{[ab]} = T \cap V(H^{ab})$ for every pair $a, b \in S$ with $ab \in E(G)$.

Combining all the aforementioned cases, we have

$$
T = S \cup \left( \bigcup_{a,b \in V(G) \setminus S \atop ab \in E(G)} T_{0}^{[ab]} \right) \cup \left( \bigcup_{a,b \in V(G), ab \in E(G) \atop |\{a,b\}\cap S|=1} T_{1}^{[ab]} \right) \cup \left( \bigcup_{a,b \in S \atop ab \in E(G)} T_{2}^{[ab]} \right),
$$

where $S \subseteq V(G)$, $T_{0}^{[ab]}$ is a 2-dominating set in $H^{ab}$ for every pair of $a$ and $b$ in $V(G) \setminus S$ with $ab \in E(G)$, $T_{1}^{[ab]}$ is a dominating set in $H^{ab}$ for every pair of $a$ and $b$ in $V(G)$ with $|\{a,b\}\cap S|=1$ and $ab \in E(G)$, and $T_{2}^{[ab]} \subseteq V(H^{ab})$ for every pair of $a$ and $b$ in $S$ with $ab \in E(G)$.

For the converse, suppose that

$$
T = S \cup \left( \bigcup_{a,b \in V(G) \setminus S \atop ab \in E(G)} T_{0}^{[ab]} \right) \cup \left( \bigcup_{a,b \in V(G), ab \in E(G) \atop |\{a,b\}\cap S|=1} T_{1}^{[ab]} \right) \cup \left( \bigcup_{a,b \in S \atop ab \in E(G)} T_{2}^{[ab]} \right),
$$

where $S \subseteq V(G)$, $T_{0}^{[ab]}$ is a 2-dominating set in $H^{ab}$ for every pair of $a$ and $b$ in $V(G) \setminus S$ with $ab \in E(G)$, $T_{1}^{[ab]}$ is a dominating set in $H^{ab}$ for every pair of $a$ and $b$ in $V(G)$ with $|\{a,b\}\cap S|=1$ and $ab \in E(G)$, and $T_{2}^{[ab]} \subseteq V(H^{ab})$ for every pair of $a$ and $b$ in $S$ with $ab \in E(G)$.
where $S$, $T_0^{[ab]}$, $T_1^{[ab]}$, and $T_2^{[ab]}$ each satisfy the given properties. To show that $\langle T \rangle_w$ is connected, let $x, y$ be two distinct vertices in $\langle T \rangle_w$; we will demonstrate that there is a path from $x$ to $y$ in $\langle T \rangle_w$. Consider the following cases.

Case 1. $x, y \in V(\langle \{s, t\} \cup V(H^{st}) \rangle)$ where $st \in E(G)$ for some $s, t \in V(G)$. Then in this case exactly one of the three possibilities $[x = s, y]$, $[x, t = y]$, or $[x, s, y]$ is a path that joins $x$ to $y$ in $\langle T \rangle_w$.

Case 2. $x \in V(\langle \{s, t\} \cup V(H^{st}) \rangle)$ and $y \in V(\langle \{c, d\} \cup V(H^{cd}) \rangle)$ where $st, cd \in E(G)$ for some $s, t, c, d \in V(G)$.

Subcase 2.1 Suppose that $st$ and $cd$ are incident with a common vertex $t = c$. Then exactly one of the three possibilities $[x = t, y]$, $[x, t = y]$, or $[x, t, y]$ is a path that joins $x$ to $y$ in $\langle T \rangle_w$.

Subcase 2.2 Suppose that $st$ and $cd$ are nonadjacent edges. Then by the properties of $S$, $T_0^{[cd]}$, $T_1^{[cd]}$, and $T_2^{[cd]}$ as subsets of $T$, $T \cap V(\langle \{a, b\} \cup V(H^{ab}) \rangle)$ is a dominating set in $\langle \{a, b\} \cup V(H^{ab}) \rangle$ for every $a, b \in V(G)$ with $ab \in E(G)$. Since $G$ is connected and nontrivial, it follows that $N_{G \odot H}[T] = V(G \odot H)$. Since $N_{G \odot H}[T] = V(\langle T \rangle_w)$, we immediately obtain $V(\langle T \rangle_w) = V(G \odot H)$. Since $G$ again is connected, there is a path $P = [t = g_1, g_2, ..., g_k = c]$ in $G$ which we can easily follow in coming up with a path $P^*$ starting from $x$ and going to $y$, passing thru one vertex of $H^{g_ig_{i+1}}$ in case both $g_i$ and $g_{i+1}$ do not belong to $T$.

The two cases above show that $x$ and $y$ are connected by a path in $\langle T \rangle_w$. Hence, $\langle T \rangle_w$ is connected.

To show that $T$ is a 2-dominating set in $G \odot H$, let $x \in V(G \odot H) \setminus T$. We consider the following cases:

Case 1. Suppose $x \in V(G) \setminus S$. Since $G$ is connected and nontrivial, there exists $y \in V(G)$ such that $xy \in E(G)$. If $y \not\in S$, then $|\{x, y\} \cap S| = 0$. By the property of $T_0^{[xy]}$, we have $|N_{\langle \{x, y\} \cup V(H^{xy}) \rangle}(x) \cap T_0^{[xy]}| \geq 2$. If $y \in S$, then $|\{x, y\} \cap S| = 1$. By the property of $T_1^{[xy]}$, there exists $c \in T_1^{[xy]}$ such that $cx \in E(\langle \{x, y\} \cup V(H^{xy}) \rangle)$. Thus, $y, c \in N_{G \odot H}(x) \cap T$.

Case 2. Let $x \in V(H^{ab}) \setminus (T \cap V(H^{ab}))$ for some $a, b \in V(G)$ with $ab \in E(G)$.

Subcase 2.1 Let $a, b \not\in S$. Then by the property of $T_0^{[ab]}$, there exist $w, z \in T_0^{[ab]}$ such that $wx, zx \in E(\langle \{a, b\} \cup V(H^{ab}) \rangle)$, implying that $w, z \in N_{G \odot H}(x) \cap T$.

Subcase 2.2 Suppose $a \in S$ and $b \not\in S$. Then by the property of $T_1^{[ab]}$, there exists $t \in T_1^{[ab]}$ such that $tx \in E(\langle \{a, b\} \cup V(H^{ab}) \rangle)$. Since $ax \in E(\langle \{a, b\} \cup V(H^{ab}) \rangle)$, we have $t, a \in N_{G \odot H}(x) \cap T$.

Subcase 2.3 Suppose $a, b \in S$. Then clearly, $ax, bx \in E(\langle \{a, b\} \cup V(H^{ab}) \rangle)$, so that $a, b \in N_{G \odot H}(x) \cap T$.

The two cases above show that every vertex $x \in V(G \odot H) \setminus T$ is dominated by at least two vertices in $T$. Finally, $T$ is now a weakly connected 2-dominating set in $G \odot H$.

\begin{theorem}
Let $G$ and $H$ be nontrivial graphs of orders $m$ and $n$, respectively,
with $G$ connected. Then $\gamma_{2w}(G \diamond H) \geq m$.

**Proof:** We are going to show that every weakly connected 2-dominating set in $G \diamond H$ has cardinality at least $m$. Let $T$ be a weakly connected 2-dominating set in $G \diamond H$. By Theorem 3.3, $T$ can be expressed as

$$T = S \cup \left( \bigcup_{a,b \in V(G) \setminus S \atop ab \in E(G)} T_{0}^{[ab]} \right) \cup \left( \bigcup_{a,b \in V(G), ab \in E(G) \atop |\{a,b\} \cap S| = 1} T_{1}^{[ab]} \right) \cup \left( \bigcup_{a,b \in S \atop ab \in E(G)} T_{2}^{[ab]} \right),$$

where $S \subseteq V(G)$, $T_{0}^{[ab]}$ is a 2-dominating set in $H_{ab}$ for every pair of $a$ and $b$ in $V(G) \setminus S$ with $ab \in E(G)$, $T_{1}^{[ab]}$ is a dominating set in $H_{ab}$ for every pair of $a$ and $b$ in $V(G)$ with $|\{a,b\} \cap S| = 1$ and $ab \in E(G)$, and $T_{2}^{[ab]} \subseteq V(H_{ab})$ for every pair of $a$ and $b$ in $S$ with $ab \in E(G)$. For convenience, set some subsets of $T$ in the given expression above as follows:

$$\mathcal{A}_0 = \bigcup_{a,b \in V(G) \setminus S \atop ab \in E(G)} T_{0}^{[ab]}, \quad \mathcal{A}_1 = \bigcup_{a,b \in V(G), ab \in E(G) \atop |\{a,b\} \cap S| = 1} T_{1}^{[ab]} \quad \text{and} \quad \mathcal{A}_2 = \bigcup_{a,b \in S \atop ab \in E(G)} T_{2}^{[ab]}.$$

We then consider two simple cases for the set $S$.

Case 1. $S = V(G)$. Then $\mathcal{A}_0$ and $\mathcal{A}_1$ are both equal to the empty set. This implies that $T = V(G) \cup \mathcal{A}_2$. Thus, $|T| \geq m$.

Case 2. $S \subsetneq V(G)$. Let $x \in V(G) \setminus S$. Since $G$ is connected and nontrivial, there exists $y \in V(G)$ such that $xy \in E(G)$. If $y \notin S$, then $T$ has a subset $T_0^{[xy]}$ contained in $\mathcal{A}_0$ such that $T_0^{[xy]}$ is 2-dominating in $H_{xy}$. Since $|T_0^{[xy]}| \geq 1$, it follows that the absence of both $x, y \in V(G)$ from $S$, and hence from $T$, does not affect $T$ cardinality-wise since $T$ would also contain at least two elements from $V(H_{xy})$. On the other hand, if $y \in S$, then $T$ has a subset $T_1^{[xy]}$ contained in $\mathcal{A}_1$ such that $T_1^{[xy]}$ is dominating in $H_{xy}$. Since $|T_1^{[xy]}| \geq 1$ in this subcase, it follows that relative to the edge $xy \in E(G)$, where $y \in S$, the absence of $x \in V(G)$ from $S$, and hence from $T$, is compensated by the fact that $T$ contains at least one element from $V(H_{xy})$. Since by Remark 3.1 the vertex sets of the various copies of $H$ as viewed in $G \diamond H$ are pairwise disjoint, it follows now from the two subcases, all for the case when $V(G) \setminus S \neq \emptyset$, that $|T| \geq |V(G)| = m$. In either case 1 or case 2, we obtain $\gamma_{2w}(G \diamond H) \geq m$. $\Box$

**Corollary 3.5** If $G$ and $H$ are nontrivial graphs where $G$ is connected, then the weakly connected 2-domination number of $G \diamond H$ is given by $\gamma_{2w}(G \diamond H) = |V(G)|$.

**Proof:** This is a direct consequence of both Theorem 3.2 and Theorem 3.4. $\Box$

Another important consequence of Theorems 3.2 and 3.4 is given next.
Corollary 3.6 Let $G$ and $H$ be nontrivial graphs with $G$ connected. Then
\[ \gamma_{2w}(G \diamond H) = 2 \] if and only if $G = K_2$.

Generally, the edge corona $G \diamond H$ may contain two or more $\gamma_{2w}$-sets. But what happens if $H$ contains no spanning star and that $\gamma_2(H) \geq 3$, such as in the case of $G \diamond H$ where $G$ is nontrivial and connected and $H$ is the path $P_n$, $n \geq 4$? The answer for this is given in the statement below, which is an immediate consequence of Theorem 3.3.

Corollary 3.7 Let $G$ be a nontrivial connected graph and $H$ be such that $\gamma(H) \geq 2$ and $\gamma_2(H) \geq 3$. Then the set $V(G)$ is the only $\gamma_{2w}$-set in $G \diamond H$.

Corollary 3.7 actually provides a realization that given two positive integers $m$ and $n$, with $m \geq 2$ and $n \geq 4$, there exist graphs $G$ and $H$ with orders $m$ and $n$, respectively, such that $V(G)$ is the only $\gamma_{2w}$-set in $G \diamond H$.

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References


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