Trees with the Total Domination Number Twice the Distance-2 Domination Number

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Abstract

The distance between two vertices u and v in a graph equals the length of a shortest path from u to v. The distance-2 domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a vertex subset where every vertex not belonging to the set is within distance two from some element of the set. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a vertex subset where every vertex of G is adjacent to an element of the set. For any nontrivial connected graph G, we can see that $\gamma_t(G) \geq 2$.

Here we focus on the trees. For $n \geq 1$, let $\mathcal{T}(n)$ be the set of trees $T$ satisfying $\gamma_t(T) = 2\gamma_2(T) = 2n$. In this paper, we provide a constructive characterization of $\mathcal{T}(n)$ for all $n \geq 1$.

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1 Introduction

One of the fastest growing areas within graph theory is the study of domination and related subset problems. The literature on the domination parameters in graphs has been detailed in the two books ([14],[15]). The decision problem of determining the domination number of a graph G is NP-complete even if

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G is bipartite [14]. We want to study two domination parameters, including distance-2 domination and total domination.

The concept of distance dominating set was initiated by Slater [20]. The distance domination problem is NP-complete for general graphs [5]. The distance dominating set is an important property used in the allocation of finite resources to massively parallel architecture. It also helps in sharing resources amongst the nodes and thereby lays the framework for designing alternate parallel paths should one or more of the nodes fail. Here we discuss the distance-2 domination. A set $D$ of vertices is a distance-2 dominating set (D2DS) if every vertex not belonging to $D$ is within distance two from some element of $D$. The distance-2 domination number of a graph $G$, denoted by $\gamma_2(G)$, is the minimum cardinality of a distance-2 dominating set in $G$. A D2DS $D$ of $G$ is called a $\gamma_2$-set of $G$ if $|D| = \gamma_2(G)$. Sridharan, Subramanian and Elias [21] obtain various upper bounds for $\gamma_2(G)$ and characterize the classes of graphs attaining these bounds. Bibi, Lakshmi and Jothilakshmi [3] presented an algorithm for finding a minimal and minimum distance-2 dominating sets of graph. They also explored on the applications of distance-2 dominating sets in networks [4].

A set $D$ of vertices in a graph $G$ is called a total dominating set (TDS) if every vertex of $G$ is adjacent to an element of $D$. The total domination number of a graph $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set in $G$. For any nontrivial connected graph $G$, we can see that $\gamma_t(G) \geq 2$. A TDS $D$ of $G$ is called a $\gamma_t$-set of $G$ if $|D| = \gamma_t(G)$. A $\gamma_t$-set $D$ of $G$ is called a ($\gamma_t, -L$)-set of $G$ if $D$ contains no leaves of $T$. The problem of determining the total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [8]. The problem of determining the total domination in graphs is now well studied in graph theory ([1],[2],[7],[9],[10],[11],[12],[13],[16],[18]). Laskar, Pfaff, Hedetniemi and Hedetniemi [19] presented a linear time algorithm to determine minimum total dominating sets of a tree and showed that for undirected path graphs the problem remains NP-complete. More details on the total domination problems can be found in survey papers by Henning [17].

For $n \geq 1$, let $\mathcal{T}(n)$ be the set of trees $T$ satisfying $\gamma_t(T) = 2\gamma_2(T) = 2n$. In this paper, we provide a constructive characterization of $\mathcal{T}(n)$ for all $n \geq 1$.

## 2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. The (open) neighborhood $N_G(v)$ of a vertex $v$ is the set of vertices adjacent to $v$ in $G$, and the closed neighborhood $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v \in A} N_G(v)$ and $N_G[A] = \bigcup_{v \in A} N_G[v]$. The degree of $v$ is the cardinality of $N_G(v)$, denoted by
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A vertex $v$ is an isolated vertex if $\deg_G(v) = 0$. A vertex $x$ is said to be a leaf if $\deg_G(x) = 1$. A vertex of $G$ is a support vertex if it is adjacent to a leaf in $G$. We denote by $L(G)$, and $U(G)$ the collections of the leaves and support vertices of $G$, respectively. For two sets $A$ and $B$, the difference of $A$ and $B$, denoted by $A - B$, is the set of all the elements of $A$ that are not elements of $B$. For a subset $A \subseteq V(G)$, the deletion of $A$ from $G$ is the graph $G - A$ obtained by removing all vertices in $A$ and all edges incident to these vertices. For a subset $B \subseteq E(G)$, the edge-deletion of $B$ from $G$ is the graph $G - B$ obtained by removing all edges in $B$ from $G$. A $u$-$v$ path $P : u = v_1, v_2, \ldots, v_k = v$ of $G$ is a sequence of $k$ vertices in $G$ such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k - 1$. For any two vertices $u$ and $v$ in $G$, the distance between $u$ and $v$, denoted by $\text{dist}_G(u, v)$, is the minimum length of the $u$-$v$ paths in $G$. Denote by $P_n$ a $n$-path with $n$ vertices. The length of $P_n$ is $n-1$. The diameter of a graph $G$ is $\text{diam}(G) = \text{max}\{\text{dist}_G(u, v) : u, v \in V(G)\}$. For any vertex $v$ of a graph $G$, the distance-2 closed neighborhood of $v$ is $N^2_G[v] = \{u : \text{dist}_G(u, v) \leq 2\}$ and the distance-2 (open) neighborhood of $v$ is $N^2_G(v) = N^2_G[v] - \{v\}$. For any subset $A \subseteq V(G)$, denote $N^2_G(A) = \bigcup_{v \in A} N^2_G(v)$ and $N^2_G[A] = \bigcup_{v \in A} N^2_G[v]$. A forest is a graph with no cycles, and a tree is a connected forest. A complete bipartite graph $K_{m,n}$ is a graph whose vertices can be divided into two disjoint and independent sets $A$ and $B$, where $|A| = m \geq 1$ and $|B| = n \geq 1$, such that every vertex of $A$ is adjacent to every vertex of $B$. The star $S(v_1, k)$ is the graph $K_{1,k-1}$, where $v_1$ is a center and $V(S(v_1, k)) = \{v_1, v_2, \ldots, v_k\}$ for $k \geq 2$. The double-star $S(v_1, v_2, k)$ is the graph consisting of the union of $S(v_1, k_1)$ and $S(v_2, k_2)$ together with an edge $v_1 v_2$, where $k = k_1 + k_2 \geq 4$. The induced subgraph $\prec A \succ_G$ induced by $A \subseteq V(G)$ is the graph with vertex set $A$ and the edge set $E(\prec A \succ_G) = \{uv \in E(G) : u, v \in A\}$. The union $G = G_1 \cup G_2$ is the graph with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2)$. For other undefined notions, the reader is referred to [6] for graph theory.

We begin with the following two straightforward observations.

**Observation 2.1.** If $D$ is a D2DS of a graph $G$, then $N^2_G[D] = V(G)$.

**Observation 2.2.** If $D$ is a TDS of a graph $G$, then $N_G(D) = V(G)$.

The following lemmas are useful.

**Lemma 2.3.** Let $T$ be a nontrivial tree. If $D$ is a TDS of $T$, then $\prec D \succ_T$ has no isolated vertex.

*Proof.* If $v$ is an isolated vertex of the subgraph $\prec D \succ_T$, then $N_T(v) \cap D = \emptyset$ and $v \notin N_T(D)$. So $D$ is not a TDS of $T$, this is a contradiction. $\Box$

**Lemma 2.4.** Let $T$ be a tree with at least four vertices. If $T$ is not a star, then there exists a $(\gamma_t,-L)$-set of $T$. 

This contradicts that \( D \cap L(T) = \emptyset \), then we are done. So we assume that \( D \cap L(T) = \{x_1, \ldots, x_k\} \), where \( k \geq 1 \). Let \( y_i \in N_T(x_i) \), where \( i = 1, \ldots, k \). By Lemma 2.3, \( y_i \in D \) for \( i = 1, \ldots, k \). Since \( T \) is not a star, there exists a vertex \( z_i \notin L(T) \) satisfying \( dist_T(z_i, x_i) = 2 \), where \( i = 1, \ldots, k \). If \( z_i \in D \) for some \( i \), then \( D - \{x_i\} \) is a TDS of \( T \) with cardinality \( |D| - 1 \). This contradicts that \( D \) is a \( \gamma_t \)-set of \( T \), so \( z_i \notin D \) for all \( i \). Let \( D^* = (D - \{x_1, \ldots, x_k\}) \cup \{z_1, \ldots, z_k\} \). Note that \( N_T(\{x_1, \ldots, x_k\}) \subseteq N_T(\{z_1, \ldots, z_k\}) \) and \( y_i \in D \) for \( i = 1, \ldots, k \). Then \( N_T(D^*) = V(T) \), thus \( D^* \) is a TDS with cardinality \( |D^*| \leq |D| \). Then \( \gamma_t(T) \leq |D^*| \leq |D| = \gamma_t(T) \), so \( |D^*| = \gamma_t(T) \). Hence \( D^* \) is a \( (\gamma_t, -L) \)-set of \( T \).

**Lemma 2.5.** Let \( T \) be a tree with at least two vertices. If \( x \) is a leaf of \( T \), then \( \gamma_2(T - \{x\}) \leq \gamma_2(T) \).

**Proof.** Let \( T' = T - \{x\} \). Suppose \( S \) is a \( \gamma_2 \)-set of \( T \), we consider two cases.

**Case 1.** \( x \notin S \). Then \( S \) is a distance-2 dominating set of \( T' \), so \( \gamma_2(T') \leq |S| = \gamma_2(T) \).

**Case 2.** \( x \in S \). Let \( y \) be the support vertex adjacent to \( x \) in \( T \). Since \( S \) is a \( \gamma_2 \)-set of \( T \), this means that \( y \notin S \). If \( v \neq x \) is a vertex of \( T \), then \( dist_T(v, y) \leq dist_T(v, x) - 1 \). Thus \( S' = (S - \{x\}) \cup \{y\} \) is a D2DS of \( T' \), so \( \gamma_2(T') \leq |S'| = |S| = \gamma_2(T) \).

By Case 1 and Case 2, we have \( \gamma_2(T - \{x\}) \leq \gamma_2(T) \).

**Lemma 2.6.** Suppose \( T \in \mathcal{F}(n) \) and \( S = \{v_1, u_1, v_2, u_2, \ldots, v_n, u_n\} \) is a \( \gamma_t \)-set of \( T \), where \( v_iu_i \in E(T) \) for \( i = 1, \ldots, n \). Let \( S_1 = \{v_1, \ldots, v_n\} \) and \( S_2 = \{u_1, \ldots, u_n\} \). Then \( S_i \) is a DDS of \( T \), where \( i = 1, 2 \).

**Proof.** Let \( A_i = N_T[S_i] \) for \( i = 1, 2 \). If there exists a vertex \( v \notin A_i \), then \( v \notin N_T[S_i] \). So \( N_T[S] \neq V(T) \), this is a contradiction. Hence \( A_i = V(T) \) for \( i = 1, 2 \). Hence \( S_i \) is a D2DS of \( T \), where \( i = 1, 2 \).

## 3 Characterization

In this section, we characterize the set \( \mathcal{F}(n) \) for all \( n \geq 1 \).

**Lemma 3.1.** The set \( \mathcal{F}(1) \) is the collection of all the stars and double-stars.

**Proof.** Let \( T \in \mathcal{F}(1) \). Then \( \gamma_t(T) = 2 \) and \( \gamma_2(T) = 1 \). If \( \text{diam}(T) \geq 5 \), then \( \gamma_t(T) \geq \gamma_t(P_5) \geq 3 \). This is a contradiction, so \( \text{diam}(T) \leq 4 \). Hence \( T \) is a star or double-star.

In order to give a constructive characterization of \( \mathcal{F}(n) \), we introduce the following operations and the vertex subsets \( \mathcal{A}^1(T'), \mathcal{A}^2(T') \) and \( \mathcal{B}(T') \), where \( T' \) is a tree.

\[
\mathcal{A}^1(T') = \{u : \gamma_2(T' - \{u\}) < \gamma_2(T')\}.
\]
\[ A^2(T') = \{ u : \gamma_2(T' - N_{T'}[u]) < \gamma_2(T') \} \]
\[ B(T') = \{ u : u \in S \text{ for some } \gamma_r\text{-set of } T' \}. \]

**Operation O1.** Assume \( u \notin A^1(T') \), add a double star \( T^* = S(w_1, w_2, k) \) and the edge \( uw' \), where \( k \geq 4 \) and \( w' \notin \{ w_1, w_2 \} \) (see Figure 1).

![Figure 1: The Operations O1 and O2](image)

**Operation O2.** Assume \( u \notin A^2(T') \), add a double star \( T^* = S(w_1, w_2, k) \), where \( k \geq 4 \), and the edge \( uw_1 \) (see Figure 1).

**Operation O3.** Assume \( u \notin A^2(T') \), add a star \( T^* = S(w_1, k) \) and the edge \( uw' \), where \( k \geq 3 \) and \( w' \neq w_1 \) (see Figure 2).

![Figure 2: The Operations O3 and O4](image)

**Operation O4.** Assume \( u \notin B(T') \), add a star \( T^* = S(w_1, k) \), where \( k \geq 2 \), and the edge \( uw_1 \) (see Figure 2).

Let \( \Psi \) be the collection of the trees \( T \) which are obtained from a sequence \( T_1 \in \mathcal{S}(1), T_2, \ldots, T_{k-1}, T_k = T \) and, if \( i = 1, 2, \ldots, k-1, T_{i+1} \) can be obtained recursively from \( T_i \) by one of the Operations O1-O4. Suppose that \( \Psi(n) \) is the collection of the trees \( T \) in \( \Psi \) satisfying \( \gamma_r(T) = 2\gamma_2(T) = 2n \). By Lemma 3.1, we can see that \( \mathcal{S}(1) = \Psi(1) \). We want to prove \( \mathcal{S}(n) = \Psi(n) \) for all \( n \geq 1 \). Theorem 3.2 is the main theorem.

**Theorem 3.2.** For \( n \geq 1 \), \( \mathcal{S}(n) = \Psi(n) \).
For every tree $T \in \Psi(n)$, we can see that $\gamma_l(T) = 2\gamma_2(T) = 2n$ and $T \in \mathcal{F}(n)$. Thus $\Psi(n) \subseteq \mathcal{F}(n)$ for all $n \geq 1$. On the other hand, We will prove $\mathcal{F}(n) \subseteq \Psi(n)$ in the following lemma.

**Lemma 3.3.** For $n \geq 1$, $\mathcal{F}(n) \subseteq \Psi(n)$.

**Proof.** By Lemma 3.1, it’s true for $n = 1$. Suppose, by contradiction, $T \in \mathcal{F}(n)$ and $T \notin \Psi(n)$ such that $n$ is as small as possible. Then $n \geq 2$. Let $T'$ be a subtree of $T$ such that $T' \in \mathcal{F}(n-1)$. By the hypothesis, $T' \in \Psi(n-1)$.

Suppose $T - \{uw_0\} = T' \cup T^*$, where $T^*$ is a tree and $u \in V(T')$. Let $k = \max\{d_{T^*}(v, v_0) : v \in V(T^*)\}$. Suppose $P^* = v_0, v_1, \ldots, v_k$ is the $v_0$-$v_k$ path in $T^*$, where $v_k \in L(T^*)$. We consider the following cases.

**Case 1.** $k \geq 5$.

Let $S$ be $\gamma_l$-set of $T$ such that $\{v_{k-2}, v_{k-1}\} \subseteq S$ and $T - \{v_{k-3}v_{k-2}\} = T'' \cup T^*$. Suppose $S_1 = S \cap V(T'')$ and $S_2 = S \cap V(T^*)$, then $|S_2| = m \geq 2$ and $|S_1| = 2n - m$. Since $v_1 \notin N_T[u]$ and $v_1 \notin N_T[v_{k-2}]$, we have $|S_1| \geq \gamma_1(T') + 1 = 2(n-1) + 1 = 2n - 1$. Thus $2n = \gamma_l(T) = |S| = |S_1| + |S_2| \geq m + 2n - 1 \geq 2 + 2n - 1 = 2n + 1$. This is a contradiction.

**Case 2.** $k = 4$.

Let $S$ be $\gamma_l$-set of $T$ such that $\{v_2, v_3\} \subseteq S$ and $S' = S - \{v_2, v_3\}$. Since $d_{T'}(v_0, v_2) = 2 > 1$ and $\gamma_l(T') = 2n - 2$, we got that $u \in S$ and $deg_T(v_0) = deg_{T'}(v_1) = 2$. Let $T - \{v_0v_1\} = T'' \cup T^*$. Then $T''$ is a tree. Since $u \in S$ and $v_0 \in L(T'')$, we got $S'$ as a $\gamma_l$-set of $T''$. Since $v_0 \in L(T'')$, by Lemma 2.5, $\gamma_2(T'') \geq \gamma_2(T'' - \{v_0\}) = \gamma_2(T') = n - 1$. Since $k = 4$, we have $\gamma_2(T'') \leq \gamma_2(T) - 1 = n - 1$. So $n - 1 \geq \gamma_2(T'') \geq n - 1 = \gamma_2(T')$. Thus $\gamma_2(T'') = n - 1$, so $T'' \in \mathcal{F}(n-1)$.

By the hypothesis, $T'' \in \Psi(n-1)$. Since $\gamma_2(T'') = \gamma_2(T' - \{v_0\})$, this means that $v_0 \notin A^1(T'')$. Thus $T$ can be obtained from $T''$ by the Operation O1, so $T \in \Psi(n)$. This is a contradiction.

**Case 3.** $k = 3$.

Since $T \notin \Psi(n)$ and $T' \in \Psi(n-1)$, by the Operation O1, $u \in A^1(T')$. So $\gamma_2(T' - \{u\}) \leq \gamma_2(T') - 1 = n - 2$. Let $S''$ be a $\gamma_l$-set of $T' - \{u\}$. Then $S = S'' \cup \{v_1\}$ is a D2DS of $T$, so $n = \gamma_2(T) \leq |S| = |S''| + 1 = \gamma_2(T'') + 1 \leq (n - 2) + 1 = n - 1$. This is a contradiction.

**Case 4.** $k = 2$.

Since $T \notin \Psi(n)$ and $T' \in \Psi(n-1)$, by the Operation O2 and O3, $u \in A^2(T')$. So $\gamma_2(T' - N_T[u]) \leq \gamma_2(T') - 1 = n - 2$. Let $S''$ be a $\gamma_l$-set of $T' - N_T[u]$. Then $S = S'' \cup \{v_0\}$ is a D2DS of $T$, so $n = \gamma_2(T) \leq |S| = |S''| + 1 = \gamma_2(T'') + 1 \leq (n - 2) + 1 = n - 1$. This is a contradiction.

**Case 5.** $k = 1$.

Since $T \notin \Psi(n)$ and $T' \in \Psi(n-1)$, by the Operation O4, $u \in B(T')$. So $u \in S'$ for some $\gamma_l$-set $S'$ of $T'$. Then $S = S' \cup \{v_0\}$ is a TDS of $T$. Thus $2n = \gamma_l(T) \leq |S| = |S'| + 1 = \gamma_l(T') = 2n - 2 + 1 = 2n - 1$. This is a contradiction.
Case 6. $k = 0$.

Then $v \in L(T)$. Let $S'$ be a $\gamma_t$-set of $T'$. By Lemma 2.6, there exists a D2DS $D' \subset S'$, where $|D'| = \frac{1}{2}|S'|$, such that $u \in N_{T'}[D']$. Suppose $u' \in D'$ and $u' \in N_{T'}[u]$, then $\text{dist}_{T}(v_0, u') \leq 2$, so $D'$ is a D2DS of $T$. Thus $n = \gamma_2(T) \leq |D'| = \gamma_2(T') = n - 1$. This is a contradiction.

By Case 1, Case 2, Case 3, Case 4, Case 5 and Case 6, which complete the proof. 

As an immediate consequence of Lemma 3.3, we obtain the Theorem 3.2.

Hence we provide a constructive characterization $\Psi(n)$ of $\mathcal{T}(n)$ for all $n \geq 1$.

References


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