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Weak-Trees with Equal Distance-2 Domination and Independent Domination

Guo-Chin Gao, Min-Jen Jou [1] and Jenq-Jong Lin

Ling Tung University, Taichung 40852, Taiwan

Abstract

The distance between two vertices u and v in a graph equals the length of a shortest path from u to v. The distance-2 domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a vertex subset where every vertex not belonging to the set is within distance two from some element of the set. The independent domination number of a graph G, denoted by $\gamma_i(G)$, is the minimum cardinality of a vertex subset that it is an independent set and every vertex not belonging to the set is adjacent to an element of the set. A weak-tree is a tree which has no duplicated leaf. Here we focus on the weak-trees. Let $\mathscr{T}(n)$ be the set of weak-trees $\widetilde{T}$ satisfying $\gamma_2(\widetilde{T}) = \gamma_i(\widetilde{T}) = n$. In this paper, we provide a constructive characterization of $\mathscr{T}(n)$ for all $n \geq 2$.

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Keywords: weak-tree, distance-2 domination number, independent domination number

1 Introduction

The study of domination and related subset problems is one of the fastest growing areas within graph theory. The literature on the domination parameters in graphs has been detailed in the two books ([6],[7]). The decision problem of determining the domination number of a graph G is NP-complete even if

[1]Corresponding author

G is bipartite [6]. We want to study two domination parameters, including distance-2 domination and independent domination.

The concept of distance dominating set was initiated by Slater [8]. The distance domination problem is NP-complete for general graphs [3]. The distance dominating set is an important property used in the allocation of finite resources to massively parallel architecture. It also helps in sharing resources amongst the nodes and thereby lays the framework for designing alternate parallel paths should one or more of the nodes fail. Here we consider the distance-2 domination. A set S of vertices is a *distance-2 dominating set* (D2DS) if every vertex not belonging to S is within distance two from some element of S. The *distance-2 domination number* of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a distance-2 dominating set in G. A D2DS S of G is called a γ_2*-set* of G if $|S| = \gamma_2(G)$. Sridharan, Subramanian and Elias [9] obtain various upper bounds for $\gamma_2(G)$ and characterize the classes of graphs attaining these bounds. Bibi, Lakshmi and Jothilakshmi [1] presented an algorithm for finding a minimal and minimum distance-2 dominating sets of graph. They also explored on the applications of distance-2 dominating sets in networks [2].

A set I of vertices is an *independent dominating set* (IDS) if I is an independent set and every vertex not belonging to I is adjacent to an element of I. The *independent domination number* of a graph G, denoted by $\gamma_i(G)$, is the minimum cardinality of a IDS in G. A IDS I of G is called a γ_i*-set* of G if $|I| = \gamma_i(G)$. W. Goddard and M.A. Henning [5] offer a survey of selected recent results on independent domination in graphs.

For $n \geq 2$, let $\mathscr{T}(n)$ be the set of weak-trees $\widetilde{T}$ satisfying $\gamma_2(\widetilde{T}) = \gamma_i(\widetilde{T}) = n$. In this paper, we provide a constructive characterization of $\mathscr{T}(n)$ for all $n \geq 2$.

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v\in A} N_G(v)$ and $N_G[A] = \bigcup_{v\in A} N_G[v]$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex x is said to be a *leaf* of G if $\deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. We denote by $L(G)$, and $U(G)$ the collections of the leaves and support vertices of G, respectively. For two sets A and B, the *difference of* A *and* B, denoted by $A-B$, is the set of all the elements of A that are not elements of B. For a subset $A \subseteq V(G)$, the *deletion of* A *from* G is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. For a subset $B \subseteq E(G)$, the *edge-deletion of* B *from* G is the graph $G - B$ obtained by removing all edges in B from G. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$. For any two vertices u and v in G, the *distance between* u *and* v, denoted by $dist_G(u, v)$, is the minimum length of the u-v paths in G. Denote by P_n a n*-path* with n vertices. The length of P_n is n-1. The *diameter* of a graph G is $diam(G) = \max\{dist_G(u, v) : u, v \in V(G)\}$. For any vertex v of a graph G, the *distance-2 closed neighborhood of* v is $N^2_G[v] = \{u : dist_G(u, v) \leq 2\}$ and the *distance-2* (*open*) *neighborhood of* v is $N^2_G(v) = N^2_G[v]-\{v\}$. For any subset $A \subseteq V(G)$, denote $N^2_G(A) = \bigcup_{v\in A} N^2_G(v)$ and $N^2_G[A] = \bigcup_{v\in A} N^2_G[v]$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. A *weak-tree* is a tree which has no duplicated leaf. The *induced subgraph* $\prec A \succ_G$ induced by $A \subseteq V(G)$ is the graph with vertex set A and the edge set $E(\prec A \succ_G) = \{uv \in E(G) : u, v \in A\}$. The *union* $G = G_1 \cup G_2$ is the graph with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2)$. For other undefined notions, the reader is referred to [4] for graph theory.

We begin with the following straightforward observation.

Observation 2.1. *If* S *is a D2DS of a graph* G*, then* $N^2_G[S] = V(G)$*.*

Lemma 2.2. *Suppose* T *is a tree, then* $\gamma_2(T) \leq \gamma_i(T)$*.*

Proof. If S is a γ_i-set of T, then $N_T[S] = V(T)$. So $N^2_T[S] = V(T)$, thus S is a D2DS of T. Hence $\gamma_2(T) \leq |S| = \gamma_i(T)$, we complete the proof. □

Lemma 2.3. *Suppose* $\widetilde{T}$ *is a weak-tree and* $|L(\widetilde{T})| = l$*. Then* $\gamma_i(\widetilde{T}) \geq l$*.*

Proof. Since $\widetilde{T}$ is a weak-tree and $|L(\widetilde{T})| = l$, we have that $|U(\widetilde{T})| = |L(\widetilde{T})| = l$. Let I be a γ_i-set of $\widetilde{T}$. Then $I \cap N_T[x] \neq \emptyset$ for every $x \in L(\widetilde{T})$. If x and x' are two distinct leaves of $\widetilde{T}$, then $N_{\widetilde{T}}[x] \cap N_{\widetilde{T}}[x'] = \emptyset$. So $\gamma_i(\widetilde{T}) \geq |L(\widetilde{T})| = l$, we complete the proof. □

Lemma 2.4. *Suppose* T *is a tree with at least three vertices, then there exists a* γ_2*-set* S *of* T *satisfying* $S \cap L(T) = \emptyset$*.*

Proof. Let S be a γ_2-set of T. If $S \cap L(T) = \emptyset$, then we are done. So we assume that $S \cap L(T) = \{x_1, \ldots, x_k\}$, where $k \geq 1$. Let $y_i \in N_T(x_i)$, where $i = 1, \ldots, k$. Then $y_i \notin S$ for all i. Let $S^* = (S - \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}$. Then S^* is a D2DS of T with cardinality $|S^*| = |S|$. Then $\gamma_2(T) \leq |S^*| = |S| = \gamma_2(T)$, so $|S^*| = \gamma_2(T)$. Hence S^* is a γ_2-set of T satisfying $S^* \cap L(T) = \emptyset$. □

3 Characterization

In this section, we characterize the set $\mathscr{T}(n)$ for all $n \geq 2$, where $\mathscr{T}(n)$ is the collection of the weak-trees $\widetilde{T}$ satisfying $\gamma_2(\widetilde{T}) = \gamma_i(\widetilde{T}) = n$. First, we characterize the set $\mathscr{T}(2)$.

Lemma 3.1. *$\mathscr{T}(2) = \{P_6\}$.*

Proof. Let $\widetilde{T} \in \widetilde{\mathscr{T}}(2)$. Since $\gamma_2(\widetilde{T}) = 2$, $diam(\widetilde{T}) \geq 5$. Since $\gamma_i(\widetilde{T}) = 2$, $diam(\widetilde{T}) \leq 5$. So $diam(\widetilde{T}) = 5$. By Lemma 2.3, $2 = \gamma_i(\widetilde{T}) \geq |L(\widetilde{T})| \geq 2$. Then $|L(\widetilde{T})| = 2$ and $\widetilde{T}$ is a path. Thus $\widetilde{T}$ is a path and $diam(\widetilde{T}) = 5$, hence $\widetilde{T} = P_6$ and $\widetilde{\mathscr{T}}(2) = \{P_6\}$. □

In order to give a constructive characterization of $\mathscr{T}(n)$, where $n \geq 2$, we introduce the tree $\mathcal{T}_n$. The tree $\mathcal{T}_n$ is a weak-tree of order $3n$, where $n \geq 2$, satisfying the following properties.

(i) $V(\mathcal{T}_n) = A(\mathcal{T}_n) \cup U(\mathcal{T}_n) \cup L(\mathcal{T}_n)$.

(ii) $\prec A(\mathcal{T}_n) \succ_{\mathcal{T}_n}$ is a tree, where $|A(\mathcal{T}_n)| = n$.

We can see that $P_6 = \mathcal{T}_2$, by Lemma 3.1, $\mathscr{T}(2) = \{P_6\} = \{\mathcal{T}_2\}$. We want to prove $\mathscr{T}(n) = \{\mathcal{T}_n\}$ for all $n \geq 2$. Theorem 3.2 is the main theorem.

Theorem 3.2. *For $n \geq 2$, $\mathscr{T}(n) = \{\mathcal{T}_n\}$.*

We can see that $\gamma_2(\mathcal{T}_n) = |A(\mathcal{T}_n)| = n$ and $\gamma_i(\mathcal{T}_n) = |U(\mathcal{T}_n)| = n$, where $n \geq 2$, thus $\mathcal{T}_n \in \mathscr{T}(n)$. On the other hand, We will prove the following lemma.

Lemma 3.3. *Suppose $\widetilde{T} \in \mathscr{T}(n)$, where $n \geq 2$, then $\widetilde{T} = \mathcal{T}_n$.*

Proof. We prove it by induction on n. By Lemma 3.1, $\mathscr{T}(2) = \{P_6\} = \{\mathcal{T}_2\}$, so it's true for $n = 2$. Assume that it's true for $n-1$, where $n \geq 3$. Let $\widetilde{T} \in \mathscr{T}(n)$ and $P : x, y, z, w, \ldots$ be a longest path of $\widetilde{T}$. Suppose the edge-deletion $\widetilde{T} - \{zw\} = T^* \cup T'$, where T^* and T' are trees. Since T' is a subtree of $\widetilde{T}$, this means that $\gamma_2(T') \geq n-1$.
Claim 1. $N_{\widetilde{T}}(z) = \{y, z\}$.

Suppose, by contradiction, $|N_{\widetilde{T}}(z)| \geq 3$. Since P is a longest path of $\widetilde{T}$, the neighbors of z in T^* are leaves or support vertices. Let $Q_1 = N_{T^*}(z) \cap U(T^*)$ and $Q_2 = N_{T^*}(z) \cap L(T^*)$. Suppose, by contradiction, $|Q_1| \geq 2$. Then $\gamma_i(T') \leq n - |Q_1| \leq n-2$. By Lemma 2.2, $n-1 \leq \gamma_2(T') \leq \gamma_i(T') \leq n-2$. This is a contradiction, so $Q_1 = \{y\}$. Suppose, by contradiction, $|Q_2| \geq 1$. Then $\gamma_i(T') \leq n - |Q_1| - |Q_2| \leq (n-1) - 1 = n-2$. By Lemma 2.2, $n-1 \leq \gamma_2(T') \leq \gamma_i(T') \leq n-2$. This is a contradiction, so $Q_2 = \emptyset$. Hence $N_{\widetilde{T}}(z) = \{y, z\}$.

By Claim 1, we can see that $T^* = P_3$ and $\gamma_i(T') \leq n-1$. And we have that $n-1 \leq \gamma_2(T') \leq \gamma_i(T') \leq n-1$, so $\gamma_2(T') = \gamma_i(T') = n-1$.
Claim 2. T' is a weak-tree.

Suppose, by contradiction, T' have duplicated leaves. Since $\widetilde{T}$ has no duplicated leaf, this means that T' have only two duplicated leaves w and w'.

Let $T'' = T' - \{w'\}$. Then T'' is a weak-tree, where $w \in L(T'')$. By Lemma 2.4, $\gamma_2(T'') = \gamma_2(T') = n-1$. Thus $n-1 = \gamma_2(T'') \leq \gamma_i(T'') \leq \gamma_i(T') = n-1$. Hence $T'' \in \mathscr{T}(n-1)$. By induction hypothesis, $T'' = \mathcal{T}_{n-1}$. Thus $V(T'') = A(T'') \cup U(T'') \cup L(T'')$, where $w \in L(T'')$, and $\prec A(T'') \succ_{T''}$ is a tree of order $|A(T'')| \geq 2$. Let $u \in A(T'')$ such that $dist_{T''}(u, w) = 2$, and let $S_1 = (A(T'') - \{u\}) \cup \{z\}$. Then $u \in N_T^2[A(T'') - \{u\}]$ and $N_{\widetilde{T}}^2[S_1] = V(\widetilde{T})$. Then S_1 is a D2DS of $\widetilde{T}$ with $|S_1| = n-1$. This is a contradiction, so T' is a weak-tree.

By Claim 2, $T' \in \mathscr{T}(n-1)$. By induction hypothesis, $T' = \mathcal{T}_{n-1}$. Thus $V(T') = A(T') \cup U(T') \cup L(T')$ and $\prec A(T') \succ_{T'}$ is a tree of order $|A(T')| \geq 2$.

Claim 3. $w \in A(T')$.

Suppose, by contradiction, $w \in U(T')$ or $w \in L(T')$. Let $v \in A(T')$ such that $dist_{T'}(u, w)$ is as small as possible. Then $S_2 = (A(T') - \{v\}) \cup \{z\}$ is a D2DS of $\widetilde{T}$ with $|S_2| = n-1$. This is a contradiction, so $w \in A(T')$.

By Claim 3, we obtain that $\widetilde{T} = \mathcal{T}_n$. □

As an immediate consequence of Lemma 3.3, we obtain the Theorem 3.2. Hence we provide that $\mathscr{T}(n) = \{\mathcal{T}_n\}$ for all $n \geq 2$.

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