Webs on $S^2$ Formed by Pencils of Planes

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Abstract

We describe a hexagonal 3-web cut off $S^2$ by three pencils of planes, which is a result of Asano. Applying its result, we give a totally hexagonal 4-web on $S^2$.

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1 Introduction

A $d$-web in a surface $S$ is a configuration of $d$ curvilinear foliations on $S$, which mutual transverse ($d \geq 3$). The simplest 3-web is three families of parallel lines, which is diffeomorphic to three families of level curves given by three functions $(x, y, x + y)$ in $\mathbb{R}^2 = \{(x, y)\}$. Such 3-webs are called hexagonal.

In [1] Asano described 3-webs cut off the unit 2-sphere $S^2$ by three pencils of planes for three straight line axes in $\mathbb{R}^3$. Applying the stereographic projection, such 3-webs on $S^2$ are transformed to 3-webs formed by circles in $\mathbb{R}^2$. It was described in [1] that such webs are hexagonal for certain 3-axes of straight lines. Applying its result, we describe 4-axes of straight lines which give a totally hexagonal 4-web (see Section 4).

Concerning 3-webs formed by circles there are many studies. Blaschke [2] proposed a problem that is to find all hexagonal 3-webs formed by three pencils of circles in the plane. Erdogan[4], Shelekhov[9], Lazareva et al.[6]
studied a classification of hexagonal circular webs. Pottmann et al. [8] gave a classification of hexagonal 3webs formed by circles on Darboux cyclides.

We shall suppose that all mappings and manifolds are of class $C^\infty$, unless otherwise stated.

## 2 Basic results of web geometry

We review basic results of web geometry (cf. [2], [3]). Let $W = (F_1, \ldots, F_d)$ be a curvilinear $d$-web on a surfaces $S$, where $S$ is connected and oriented. We assume that $F_i$ is defined by a 1-form $\omega_i$ such that $\omega_i \wedge \omega_j \neq 0$ ($i \neq j$) for $i = 1, \ldots, d$. $W$ is diffeomorphic to $W' = (F'_1, \ldots, F'_d)$ on $S'$ if there exists diffeomorphism of $S$ to $S'$ sending $F_i$ to $F'_i$. A $d$-web $W$ is said to be linearizable if $W$ is diffeomorphic to a $d$-web consisting of $d$-foliations of straight lines.

Let $W = (\omega_1, \omega_2, \omega_3)$ be a 3-web on some domain in the plane. We may assume that

$$\omega_1 + \omega_2 + \omega_3 = 0.$$ 

Then there exists unique 1-form $\theta$ such that

$$d\omega_i = \theta \wedge \omega_i$$

(1)

for any $i$ ($i = 1, 2, 3$). The 1-form $\theta$ depends on $\omega_i$, however its exterior differential $d\theta$ does not depends on $\omega_i$ ($i = 1, 2, 3$). The 2-form $d\theta$ is called the web curvature form of $W$. A 3-web $W$ is called hexagonal if $d\theta = 0$. A hexagonal 3-web is locally diffeomorphic to the 3-web of parallel straight lines.

Let $W = (\omega_1, \omega_2, \omega_3)$ be a 3-web on some domain in $\mathbb{R}^2 = \{(x, y)\}$, where $\omega_i = a_i dx + b_i dy$ ($i = 1, 2, 3$). Let $(c_1, c_2, c_3)$ be the vector product of $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$. Then we have $c_1 \omega_1 + c_2 \omega_2 + c_3 \omega_3 = 0$.

We now set $\theta = p\omega_1 + q\omega_2$. Then by the equation (1) we have

$$p = \left( \frac{\partial (c_2 b_2)}{\partial x} - \frac{\partial (c_2 a_2)}{\partial y} \right) / c_1 c_2 c_3, \quad q = \left( -\frac{\partial (c_1 b_1)}{\partial x} + \frac{\partial (c_1 a_1)}{\partial y} \right) / c_1 c_2 c_3.$$

By direct calculations we get

**Lemma 2.1.** The web curvature form $d\theta$ is given by the following:

$$d\theta = \left( \frac{\partial}{\partial x} (pc_1 b_1 + qc_2 b_2) - \frac{\partial}{\partial y} (pc_1 a_1 + qc_2 a_2) \right) dx \wedge dy.$$

An Abelian relation of $d$-web $W = (\omega_1, \ldots, \omega_d)$ is a family of $d$ 1-forms $(\eta_1, \ldots, \eta_d)$ satisfying $d\eta_i = 0$, $\omega_i \wedge \eta_i = 0$ ($i = 1, \ldots, d$) and $\eta_1 + \cdots + \eta_d = 0$. Denote by $A(W)$ the $\mathbb{R}$-vector space of Abelian relations. The dimension of $A(W)$, called the rank of $W$, is less than or equal to $\frac{1}{2}(d - 1)(d - 2)$. A 3-web $W$ is hexagonal if and only if the rank of $W$ is maximal 1.
3 3-webs cut off $S^2$ by three pencils of planes with 3-axis

According to Asano [1], we consider the following manifolds of 3-axes composed of three projective lines $(l_1, l_2, l_3)$ in the real projective 3-space $\mathbb{P}^3$.

$X_8 = \{(l_1, l_2, l_3) \mid l_1 \cap l_2 \cap l_3$ is only one point, and all $l_i$ lie in the same plane.\}$

$X_9_I = \{(l_1, l_2, l_3) \mid l_1 \cap l_2 \cap l_3$ is only one point, and $l_i$ does not lie in the plane spanned by the other lines $l_j, l_k (i, j, k$ are all distinct).\}$

$X_9_{II} = \{(l_1, l_2, l_3) \mid \{l_i \cap l_j\} (1 \leq i < j \leq 3$ are distinct three points.\}$

$X_{10} = \{(l_1, l_2, l_3) \mid$ Two of $\{l_i \cap l_j\} (1 \leq i < j \leq 3$ are both only one point, and the others are empty.\}$

$X_{11} = \{(l_1, l_2, l_3) \mid$ One of $\{l_i \cap l_j\} (1 \leq i < j \leq 3$ is only one point, and the others are empty.\}$

$X_{12} = \{(l_1, l_2, l_3) \mid \{l_i \cap l_j\} (1 \leq i < j \leq 3$ are all empty.\}$

The subscript $n$ of $X_n$ is the dimension of $X_n$.

We decompose $\mathbb{P}^3 = \{[x_0 : x_1 : x_2 : x_3]\} \to \mathbb{R}^3 = \{(x, y, z)\}$ and the hyperplane at infinity $\infty^2 = \{[0 : x_1 : x_2 : x_3]\}$. The figures in Figure 1 show $(l_1, l_2, l_3) \in X_n$ with the case $l_i \cap l_j \notin \infty^2 (i < j)$.

![Figure 1: 3-axes in $X_n$](image)

The affine part of the pencil of projective planes through a projective line $l$ in $\mathbb{P}^3$ is called the pencil of planes in $\mathbb{R}^3$ with axis $l$ and is denoted by $\Pi(l)$. Then we consider the 3-webs formed by the intersection of three pencils of planes ($\Pi(l_1), \Pi(l_2), \Pi(l_3)$) and the unit 2-sphere $S^2$ for $(l_1, l_2, l_3) \in X_n$. Such 3-webs is defined on some domain in $S^2$. Denote by $\mathcal{F}(l)$ the image of $\Pi(l) \cap S^2$ by the stereographic projection.

**Example 3.1.** Let $(l_1, l_2, l_3)$ be a 3-axis in $X_{9_I}$, which is defined as follows. The affine part of $l_1$ is $x = 1, z = 0$. Similarly $l_2 : x = -1, z = 0, l_3 : x = 0, z = 1$. These 3 lines intersect at an infinity point $[0 : 1 : 0 : 0]$. Then the
Applying the stereographic projection the image \( \mathcal{F}(l_1) \) of \( \Pi(l_1) : z = k(x - 1) \) \((k \in \mathbb{R})\) by the stereographic projection is given by the 1-form \( df_1 \), where

\[
  f_1(x, y) = \frac{1 - x^2 - y^2}{(x - 1)^2 + y^2}.
\]

Similarly \( \mathcal{F}(l_2), \mathcal{F}(l_3) \) are given by \( df_2, df_3 \), where

\[
  f_2(x, y) = \frac{1 - x^2 - y^2}{(x + 1)^2 + y^2}, \quad f_3(x, y) = x.
\]

\( \mathcal{F}(l_1), \mathcal{F}(l_2) \) are parabolic pencils of circles. Applying Lemma 2.1, by direct calculations we have \( d\theta = 0 \). That is, the 3-web cut off \( S^2 \) by \((\Pi(l_1), \Pi(l_2), \Pi(l_3))\) is hexagonal.

**Example 3.2.** Let \((l_1, l_2, l_3)\) be a 3-axis in \( X_{9H} \), which is defined as follows. The affine part of \( l_1 \) is \( x = 0, z = 1 \). Similarly \( l_2 : x + y = 1, z = 1, l_3 : x - y = 1, z = 1 \). Then \( \mathcal{F}(l_i) \) is given by the 1-form \( df_i \) \((i = 1, 2, 3)\), where

\[
  f_1(x, y) = x,
\]

\[
  f_2(x, y) = \frac{1 - 2x + x^2 - 2y + y^2}{-1 - 2x + x^2 - 2y + y^2}, \quad f_3(x, y) = \frac{1 - 2x + x^2 + 2y + y^2}{-1 - 2x + x^2 + 2y + y^2}.
\]

Applying Lemma 2.1, by direct calculations we have

\[
d\theta = \frac{-2y(1 - x)}{(1 - y^2)^2} dx \wedge dy.
\]

That is, the 3-web in this case is not hexagonal.

The following main theorem has been firstly established by Asano [1].

**Theorem 3.3.** If \((l_1, l_2, l_3) \subseteq X_8, X_{9f}\), then a 3-web formed by the intersection of \((\Pi(l_1), \Pi(l_2), \Pi(l_3))\) and \( S^2 \) is hexagonal.

**Proof.** The idea of our proof is the same as [1], however we describe more exactly.

Let \((l_1, l_2, l_3)\) be a 3-axis that belongs to \( X_8, X_{9f}\).

To begin with, consider the following case.

(I) \( l_i \cap l_j \notin \infty^2 \) \((i < j)\) : We take points \( P_0(a, 0, 0), P_1(0, b, c), P_2(0, d, e), P_3(0, f, 0) \) in \( \mathbb{R}^3 = \{(x, y, z)\} \). Denote the straight lines \( P_0P_1, P_0P_2, P_0P_3 \) by \( l_1, l_2, l_3 \) respectively. Then \((l_1, l_2, l_3)\) belongs to \( X_8, X_{9f}\).

(i) Case \( a \neq 0 \) : \( \Pi(l_1) \) is given by \((bx + ay - ab) + k(cx + az - ac) = 0 \) \((k \in \mathbb{R})\). Applying the stereographic projection the image \( \mathcal{F}(l_1) \) is given by the level curves of the following function

\[
  f_1(x, y) = \frac{ab - 2bx + abx^2 - 2ay + aby^2}{a + ac - 2cx - ax^2 + acx^2 - ay^2 + acy^2}.
\]
Similarly $\mathcal{F}(l_2), \mathcal{F}(l_3)$ are given by the following functions respectively:

$$f_2(x, y) = \frac{ad - 2dx + adx^2 - 2ay +ady^2}{a + ae - 2ex - ax^2 + aex^2 - ay^2 + aey^2},$$

$$f_3(x, y) = \frac{af - 2fx + afx^2 - 2ay + afy^2}{-1 + x^2 + y^2}.$$  

Then applying Lemma 2.1, by direct calculations (using a computer), we have $d\theta = 0$.

(ii) Case $a = 0$ : By the same argument as (i), we have

$$f_1(x, y) = \frac{2x}{2cy - b(-1 + x^2 + y^2)},$$

$$f_2(x, y) = \frac{2x}{2cy - d(-1 + x^2 + y^2)},$$

$$f_3(x, y) = \frac{2x}{-1 + x^2 + y^2}.$$  

Then applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

Next consider the following case.

(II) $l_i \cap l_j \in \infty^2 (i < j)$ : Moreover there is the cases whether one of three projective lines of $(l_1, l_2, l_3) \in X_8, X_{9f}$ lies in $\infty^2$ or not. We may suppose the infinite point $l_i \cap l_j$ is $[0 : 1 : 0 : 0]$.

(iii) Case that none of three projective lines lie in $\infty^2$ : We take points $P_1(a_1, 0, b), P_2(a_2, 0, b), P_3(a_3, 0, c)$. Denote by $l_i (i = 1, 2, 3)$ the straight line through $P_i$ with the direction vector $(0, 1, 0)$. Then $(l_1, l_2, l_3)$ belongs to $X_8$ if $b = c$ and $a_i \neq a_j (i \neq j)$, belongs to $X_{9f}$ if $b \neq c$ and $a_1 \neq a_2$. By the same argument as the above, $\mathcal{F}(l_1), \mathcal{F}(l_2), \mathcal{F}(l_3)$ are given by the following functions

$$f_i(x, y) = \frac{2x - a_i(1 + x^2 + y^2)}{-1 + x^2 + y^2 - b(1 + x^2 + y^2)} (i = 1, 2),$$

$$f_3(x, y) = \frac{2x - a_3(1 + x^2 + y^2)}{-1 + x^2 + y^2 - c(1 + x^2 + y^2)}.$$  

Applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

(iv) Case that one of three projective lines lies in $\infty^2$ : We may suppose $l_3$ lies in $\infty^2$ and both $l_1, l_2$ do not. So let $l_i (i = 1, 2)$ be the straight lines parallel to $y$-axis through the point $(a_i, 0, b_i) (a_1 \neq a_2)$. Therefore $\mathcal{F}(l_i)$ is given by

$$f_i(x, y) = \frac{2x - a_i(1 + x^2 + y^2)}{-1 + x^2 + y^2 - b_i(1 + x^2 + y^2)} (i = 1, 2).$$

Since the equation $m_2x_1 - m_1x_3 = 0, w = 0$ describes $l_3$, the pencil of planes $\Pi(l_3)$ is described by $m_2x = m_1z + k = 0 (k \in \mathbb{R})$. Hence $\mathcal{F}(l_3)$ is given by

$$f_3(x, y) = \frac{2m_2x - m_1(-1 + x^2 + y^2)}{1 + x^2 + y^2}.$$  

Applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

This completes the proof. \(\square\)
Remark 3.4. It can be verified that \( d\theta \) does not vanish identically for a
certain 3-axis in \( X_{10}, X_{11}, X_{12} \) as in Example 3.2. It is naturally conjectured
that \( d\theta \) does not vanish identically for any 3-axes except for \( X_8, X_{9I} \), which
shall be studied in a forthcoming paper.

4 4-webs cut off \( S^2 \) by four pencils of planes with 4-axis

Curvilinear 4-webs in the plane have particular structure as follows (cf.[5],[7]).
Let \( W = (F_1, F_2, F_3, F_4) \) be a 4-web on some domain in \( \mathbb{R}^2 \). A configuration
\( (F_i, F_j, F_k) \) made by extracting three from \( W \) is called a 3-subweb of \( W \), where
\( 1 \leq i, j, k \leq 4 \) are all distinct. A 4-web \( W \) is called totally hexagonal, if
3-subwebs of \( W \) are all hexagonal. Then the following result (Mayrhofer,
Reidemeister) is classically known:

If a planer 4-web is totally hexagonal, then it is linearizable and its rank is
maximal.

Therefore, it is important to investigate totally hexagonal 4-webs cut off \( S^2 \)
by pencils of planes with 4-axis. Let \( (\Pi(l_1), \Pi(l_2), \Pi(l_3), \Pi(l_4)) \) be four pencils
of planes in \( \mathbb{R}^3 \) with axes \( l_1, l_2, l_3, l_4 \). Applying Theorem 3.3, we immediately
have the following.

Theorem 4.1. A 4-web formed by \( S^2 \cap (\Pi(l_1), \Pi(l_2), \Pi(l_3), \Pi(l_4)) \) is totally
hexagonal if 3-subaxes \( (l_i, l_j, l_k) \) (\( 1 \leq i < j < k \leq 4 \)) are one of the followings:
(i) all 3-subaxes belong to \( X_8 \).
(ii) only one 3-subaxes belongs to \( X_8 \), other 3-subaxes belong to \( X_{9I} \).
(iii) all 3-subaxes belong to \( X_{9I} \).

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