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Webs on S^2 Formed by Pencils of Planes

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Abstract

We describe a hexagonal 3-web cut off S^2 by three pencils of planes, which is a result of Asano. Applying its result, we give a totally hexagonal 4-web on S^2.

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1 Introduction

A *d-web* in a surface S is a configuration of d curvilinear foliations on S, which mutual transverse ($d \geqq 3$). The simplest 3-web is three families of parallel lines, which is diffeomorphic to three families of level curves given by three functions $(x,\, y,\, x+y)$ in $\mathbb{R}^2 = \{(x, y)\}$. Such 3-webs are called hexagonal.

In [1] Asano described 3-webs cut off the unit 2-sphere S^2 by three pencils of planes for three straight line axes in $\mathbb{R}^3$. Applying the stereographic projection, such 3-webs on S^2 are transformed to 3-webs formed by circles in $\mathbb{R}^2$. It was described in [1] that such webs are hexagonal for certain 3-axes of straight lines. Applying its result, we describe 4-axes of straight lines which give a totally hexagonal 4-web (see Section 4).

Concerning 3-webs formed by circles there are many studies. Blaschke [2] proposed a problem that is to find all hexagonal 3-webs formed by three pencils of circles in the plane. Erdogan[4], Shelekhov[9], Lazareva et al.[6]

studied a classification of hexagonal circular webs. Pottmann et al.[8] gave a classification of hexagonal 3-webs formed by circles on Darboux cyclides.

We shall suppose that all mappings and manifolds are of class C^∞, unless otherwise stated.

2 Basic results of web geometry

We review basic results of web geometry (cf.[2],[3]). Let $\mathcal{W} = (F_1, \dots, F_d)$ be a curvilinear d-web on a surfaces S, where S is connected and oriented. We assume that F_i is defined by a 1-form ω_i such that $\omega_i \wedge \omega_j \neq 0\,(i \neq j)$ for $i = 1, \dots, d$. $\mathcal{W}$ is diffeomorphic to $\mathcal{W}' = (F'_1, \dots, F'_d)$ on S' if there exists diffeomorphism of S to S' sending F_i to F'_i. A d-web $\mathcal{W}$ is said to be *linearizable* if $\mathcal{W}$ is diffeomorphic to a d-web consisting of d-foliations of straight lines.

Let $\mathcal{W} = (\omega_1, \omega_2, \omega_3)$ be a 3-web on some domain in the plane. We may assume that

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

Then there exists unique 1-form θ such that

$$d\omega_i = \theta \wedge \omega_i \tag{1}$$

for any $i\,(i = 1, 2, 3)$. The 1-form θ depends on ω_i, however its exterior differential $d\theta$ does not depends on $\omega_i\,(i = 1, 2, 3)$. The 2-form $d\theta$ is called the *web curvature form* of $\mathcal{W}$. A 3-web $\mathcal{W}$ is called *hexagonal* if $d\theta = 0$. A hexagonal 3-web is locally diffeomorphic to the 3-web of parallel straight lines.

Let $\mathcal{W} = (\omega_1, \omega_2, \omega_3)$ be a 3-web on some domain in $\mathbb{R}^2 = \{(x, y)\}$, where $\omega_i = a_i dx + b_i dy\,(i = 1, 2.3)$. Let (c_1, c_2, c_3) be the vector product of (a_1, a_2, a_3) and (b_1, b_2, b_3). Then we have $c_1\omega_1 + c_2\omega_2 + c_3\omega_3 = 0$.

We now set $\theta = p\omega_1 + q\omega_2$. Then by the equation (1) we have

$$p = \left(\frac{\partial(c_2 b_2)}{\partial x} - \frac{\partial(c_2 a_2)}{\partial y}\right) / c_1 c_2 c_3,\ q = \left(-\frac{\partial(c_1 b_1)}{\partial x} + \frac{\partial(c_1 a_1)}{\partial y}\right) / c_1 c_2 c_3.$$

By direct calculations we get

Lemma 2.1. *The web curvature form $d\theta$ is given by the following:*

$$d\theta = \left(\frac{\partial}{\partial x}(pc_1 b_1 + qc_2 b_2) - \frac{\partial}{\partial y}(pc_1 a_1 + qc_2 a_2)\right) dx \wedge dy.$$

An *Abelian relation* of d-web $\mathcal{W} = (\omega_1, \dots, \omega_d)$ is a family of d 1-forms $(\eta_1, \dots, \eta_d)$ satisfying $d\eta_i = 0$, $\omega_i \wedge \eta_i = 0\,(i = 1, \dots, d)$ and $\eta_1 + \cdots + \eta_d = 0$. Denote by $\mathcal{A}(\mathcal{W})$ the $\mathbb{R}$-vector space of Abelian relations. The dimension of $\mathcal{A}(\mathcal{W})$, called the *rank* of $\mathcal{W}$, is less than or equal to $\frac{1}{2}(d-1)(d-2)$. A 3-web $\mathcal{W}$ is hexagonal if and only if the rank of $\mathcal{W}$ is maximal 1.

3 3-webs cut off S^2 by three pencils of planes with 3-axis

According to Asano [1], we consider the following manifolds of 3-axes composed of three projective lines $(l_1,\, l_2,\, l_3)$ in the real projective 3-space $\mathbb{P}^3$.

$X_8 = \{(l_1,\, l_2,\, l_3)\,|\, l_1 \cap l_2 \cap l_3$ is only one point, and all l_i lie in the same plane.$\}$

$X_{9I} = \{(l_1,\, l_2,\, l_3)\,|\, l_1 \cap l_2 \cap l_3$ is only one point, and l_i does not lie in the plane spanned by the other lines $l_j,\, l_k$ (i, j, k are all distinct).$\}$

$X_{9II} = \{(l_1,\, l_2,\, l_3)\,|\, \{l_i \cap l_j\}$ $(1 \le i < j \le 3)$ are distinct three points.$\}$

$X_{10} = \{(l_1,\, l_2,\, l_3)\,|$ Two of $\{l_i \cap l_j\}$ $(1 \le i < j \le 3)$ are both only one point, and these two points are distinct. The other of them is empty.$\}$

$X_{11} = \{(l_1,\, l_2,\, l_3)\,|$ One of $\{l_i \cap l_j\}$ $(1 \le i < j \le 3)$ is only one point, and the others are empty.$\}$

$X_{12} = \{(l_1,\, l_2,\, l_3)\,|\, \{l_i \cap l_j\}$ $(1 \le i < j \le 3)$ are all empty.$\}$

The subscript n of X_n is the dimension of X_n.

We decompose $\mathbb{P}^3 = \{[x_0 : x_1 : x_2 : x_3]\}$ into $\mathbb{R}^3 = \{(x, y, z)\}$ and the hyperplane at infinity $\infty^2 = \{[0 : x_1 : x_2 : x_3]\}$. The figures in Figure 1 show $(l_1, l_2, l_3) \in X_n$ with the case $l_i \cap l_j \notin \infty^2$ $(i < j)$.

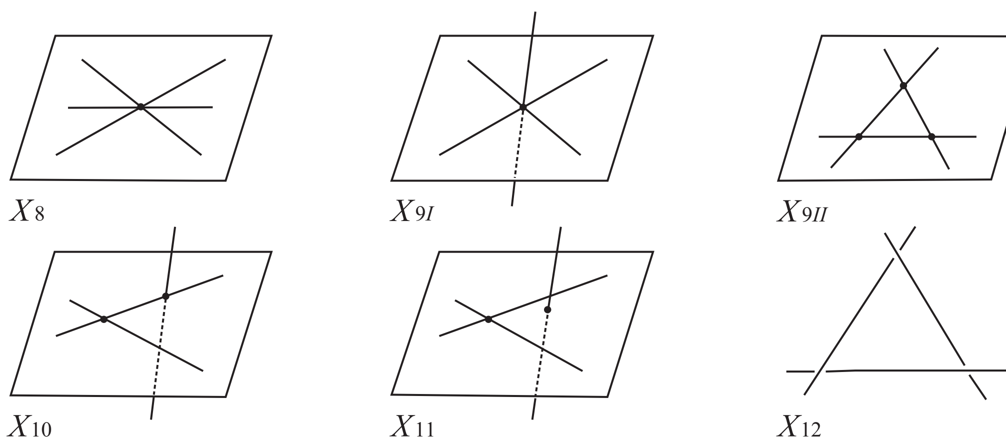

Figure 1 : 3-axes in X_n

The affine part of the pencil of projective planes through a projective line l in $\mathbb{P}^3$ is called the *pencil of planes in* $\mathbb{R}^3$ *with axis* l and is denoted by $\Pi(l)$. Then we consider the 3-webs formed by the intersection of three pencils of planes $(\Pi(l_1),\, \Pi(l_2),\, \Pi(l_3))$ and the unit 2-sphere S^2 for $(l_1, l_2, l_3) \in X_n$. Such 3-webs is defined on some domain in S^2. Denote by $\mathcal{F}(l)$ the image of $\Pi(l) \cap S^2$ by the stereographic projection.

Example 3.1. *Let* (l_1, l_2, l_3) *be a 3-axis in* X_{9I}, *which is defined as follows. The affine part of* l_1 *is* $x = 1, z = 0$. *Similarly* $l_2 : x = -1, z = 0$, $l_3 : x = 0, z = 1$. *These 3 lines intersect at an infinity point* $[0 : 1 : 0 : 0]$. *Then the* *image $\mathcal{F}(l_1)$ of $\Pi(l_1) : z = k(x-1)\,(k \in \mathbb{R})$ by the stereographic projection is given by the 1-form df_1, where*

$$f_1(x,y) = \frac{1-x^2-y^2}{(x-1)^2+y^2}.$$

Similarly $\mathcal{F}(l_2), \mathcal{F}(l_3)$ are given by df_2, df_3, where

$$f_2(x,y) = \frac{1-x^2-y^2}{(x+1)^2+y^2},\ f_3(x,y) = x.$$

$\mathcal{F}(l_1), \mathcal{F}(l_2)$ are parabolic pencils of circles. Applying Lemma 2.1, by direct calculations we have $d\theta = 0$. That is, the 3-web cut off S^2 by $(\Pi(l_1), \Pi(l_2), \Pi(l_3))$ is hexagonal.

Example 3.2. *Let (l_1, l_2, l_3) be a 3-axis in X_{9II}, which is defined as follows. The affine part of l_1 is $x = 0, z = 1$. Similarly $l_2 : x+y = 1, z = 1$, $l_3 : x-y = 1, z = 1$. Then $\mathcal{F}(l_i)$ is given by the 1-form df_i $(i = 1,2,3)$, where*

$$f_1(x,y) = x,$$
$$f_2(x,y) = \frac{1-2x+x^2-2y+y^2}{-1-2x+x^2-2y+y^2},\ f_3(x,y) = \frac{1-2x+x^2+2y+y^2}{-1-2x+x^2+2y+y^2}.$$

Applying Lemma 2.1, by direct calculations we have

$$d\theta = \frac{-2y(1-x)}{(1-y^2)^2} dx \wedge dy.$$

That is, the 3-web in this case is not hexagonal.

The following main theorem has been firstly established by Asano [1].

Theorem 3.3. *If $(l_1,\ l_2,\ l_3) \in X_8,\ X_{9I}$, then a 3-web formed by the intersection of $(\Pi(l_1),\ \Pi(l_2),\ \Pi(l_3))$ and S^2 is hexagonal.*

Proof. The idea of our proof is the same as [1], however we describe more exactly.

Let (l_1, l_2, l_3) be a 3-axis that belongs to X_8, X_{9I}.

To begin with, consider the following case.

(I) $l_i \cap l_j \notin \infty^2\,(i < j)$: We take points $\mathrm{P}_0(a,0,0)$, $\mathrm{P}_1(0,b,c), \mathrm{P}_2(0,d,e)$, $\mathrm{P}_3(0,f,0)$ in $\mathbb{R}^3 = \{(x,y,z)\}$. Denote the straight lines $\mathrm{P}_0\mathrm{P}_1, \mathrm{P}_0\mathrm{P}_2, \mathrm{P}_0\mathrm{P}_3$ by l_1, l_2, l_3 respectively. Then (l_1, l_2, l_3) belongs to X_8, X_{9I}.

(i) Case $a \neq 0$: $\Pi(l_1)$ is given by $(bx+ay-ab)+k(cx+az-ac) = 0$ $(k \in \mathbb{R})$. Applying the stereographic projection the image $\mathcal{F}(l_1)$ is given by the level curves of the following function

$$f_1(x,y) = \frac{ab-2bx+abx^2-2ay+aby^2}{a+ac-2cx-ax^2+acx^2-ay^2+acy^2}.$$

Similarly $\mathcal{F}(l_2), \mathcal{F}(l_3)$ are given by the following functions respectively:

$$f_2(x,y) = \frac{ad - 2dx + adx^2 - 2ay + ady^2}{a + ae - 2ex - ax^2 + aex^2 - ay^2 + aey^2},$$
$$f_3(x,y) = \frac{af - 2fx + afx^2 - 2ay + afy^2}{-1 + x^2 + y^2}.$$

Then applying Lemma2.1, by direct calculations (using a computer), we have $d\theta = 0$.

(ii) Case $a = 0$: By the same argument as (i), we have

$$f_1(x,y) = \frac{2x}{2cy - b(-1 + x^2 + y^2)}, f_2(x,y) = \frac{2x}{2ey - d(-1 + x^2 + y^2)},$$
$$f_3(x,y) = \frac{2x}{-1 + x^2 + y^2}.$$

Then applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

Next consider the following case.

(II) $l_i \cap l_j \in \infty^2 \, (i < j)$: Moreover there is the cases whether one of three projective lines of $(l_1, l_2, l_3) \in X_8, X_{9I}$ lies in ∞^2 or not. We may suppose the infinite point $l_i \cap l_j$ is $[0 : 1 : 0 : 0]$.

(iii) Case that none of three projective lines lie in ∞^2 : We take points $\mathrm{P}_1(a_1, 0, b)$, $\mathrm{P}_2(a_2, 0, b)$, $\mathrm{P}_3(a_3, 0, c)$. Denote by $l_i (i = 1, 2, 3)$ the straight line through P_i with the direction vector $(0, 1, 0)$. Then (l_1, l_2, l_3) belongs to X_8 if $b = c$ and $a_i \neq a_j (i \neq j)$, belongs to X_{9I} if $b \neq c$ and $a_1 \neq a_2$. By the same argument as the above, $\mathcal{F}(l_1), \mathcal{F}(l_2), \mathcal{F}(l_3)$ are given by the following functions

$$f_i(x,y) = \frac{2x - a_i(1 + x^2 + y^2)}{-1 + x^2 + y^2 - b(1 + x^2 + y^2)} \quad (i = 1, 2),$$
$$f_3(x,y) = \frac{2x - a_3(1 + x^2 + y^2)}{-1 + x^2 + y^2 - c(1 + x^2 + y^2)}.$$

Applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

(iv) Case that one of three projective lines lies in ∞^2 : We may suppose l_3 lies in ∞^2 and both l_1, l_2 do not. So let $l_i \, (i = 1, 2)$ be the straight lines parallel to y-axis through the point $(a_i, 0, b_i) \, (a_1 \neq a_2)$. Therefore $\mathcal{F}(l_i)$ is given by

$$f_i(x,y) = \frac{2x - a_i(1 + x^2 + y^2)}{-1 + x^2 + y^2 - b_i(1 + x^2 + y^2)} \quad (i = 1, 2).$$

Since the equation $m_2 x_1 - m_1 x_3 = 0, w = 0$ describes l_3, the pencil of planes $\Pi(l_3)$ is described by $m_2 x - m_1 z + k = 0 (k \in \mathbb{R})$. Hence $\mathcal{F}(l_3)$ is given by

$$f_3(x,y) = \frac{2m_2 x - m_1(-1 + x^2 + y^2)}{1 + x^2 + y^2}.$$

Applying Lemma 2.1, by direct calculations we have $d\theta = 0$.

This completes the proof. □

Remark 3.4. *It can be verified that $d\theta$ does not vanish identically for a certain 3-axis in X_{10}, X_{11}, X_{12} as in Example 3.2. It is naturally conjectured that $d\theta$ does not vanish identically for any 3-axes except for X_8, X_{9I}, which shall be studied in a forthcoming paper.*

4 4-webs cut off S^2 by four pencils of planes with 4-axis

Curvilinear 4-webs in the plane have particular structure as follows (cf.[5],[7]). Let $\mathcal{W} = (F_1, F_2, F_3, F_4)$ be a 4-web on some domain in $\mathbb{R}^2$. A configuration (F_i, F_j, F_k) made by extracting three from $\mathcal{W}$ is called a 3-*subweb* of $\mathcal{W}$, where $1 \leq i, j, k \leq 4$ are all distinct. A 4-web $\mathcal{W}$ is called *totally hexagonal*, if 3-subwebs of $\mathcal{W}$ are all hexagonal. Then the following result (Mayrhofer, Reidemeister) is classically known:

If a planer 4-web is totally hexagonal, then it is linearizable and its rank is maximal.

Therefore, it is important to investigate totally hexagonal 4-webs cut off S^2 by pencils of planes with 4-axis. Let $(\Pi(l_1), \Pi(l_2), \Pi(l_3), \Pi(l_4))$ be four pencils of planes in $\mathbb{R}^3$ with axes l_1, l_2, l_3, l_4. Applying Theorem 3.3, we immediately have the following.

Theorem 4.1. *A 4-web formed by $S^2 \cap (\Pi(l_1), \Pi(l_2), \Pi(l_3), \Pi(l_4))$ is totally hexagonal if 3-subaxes (l_i, l_j, l_k) $(1 \leq i < j < k \leq 4)$ are one of the followings:*

(i) all 3-subaxes belong to X_8.

(ii) only one 3-subaxes belongs to X_8, other 3-subaxes belong to X_{9I}.

(iii) all 3-subaxes belong to X_{9I}.

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