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Exact Solutions of Partial Differential Equations of Caputo Fractional Order

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Abstract

The integral Fourier-sine transform and integral Laplace transform were used to solve two types of partial differential equations of fractional order in xz plane, where the fractional order is the Caputo differential coefficient. The solution of the partial wave equation of fractional order was taken as a result of the first fractional partial differential equation (FPDE). The solution of Rayleigh's equation of fractional and ordinary order was taken as a result of the second fractional partial differential equation (FPDE).

Keywords: Caputo fractional derivative, Fractional derivatives, Integral transform, Rayliegh-Stokes problem, Wave equation.

Introduction

The idea of fractional calculus is as old as the traditional calculus (differentiation and integration of integer order). Leibniz (1695)**[7],** was the first scientist to discover the symbol $\frac{d^n y}{dx^n} = D^n y$ for the n th derivative, where n $\in Z^+$ (Z is integer set). Lacroix (1819) **[8,12]**, developed Leibniz's formula for n-th derivative of $y = x^m$, m is a positive integer

$$D^n y = \frac{m!}{(m-n)!} x^{m-n} \quad , \quad \text{Where } n \le m \text{ is an integer.} \qquad (1)$$

Replacing the factorial symbol by the gamma function, it further obtained the fractional derivative

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \quad , \quad \text{Since } \alpha \text{ and } \beta \text{ are fractional number} \qquad (2)$$

Joseph Liouville (1832) **,[4,9]**, formally extended the formula for the derivative of integral order n

$$D^n e^{ax} = a^n e^{ax} \qquad (3)$$

To the derivative of arbitrary order as:

$$D^\alpha e^{ax} = a^\alpha e^{ax} \qquad (4)$$

And from (4) and by using the series expansion of a function f(x), Liouville derived the formula

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x} \qquad (5)$$

Where

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x} \qquad (6)$$

Formula (5) is Liouville`s first formula for fractional derivative. The second Liouville`s formula of fractional derivative defined as

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\beta-\alpha} \qquad (7)$$

Where $\Gamma(\beta)$ is gamma function defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0 \qquad (8)$$

One of the basic properties of the gamma function is

$$z\Gamma(z) = \Gamma(z+1) \qquad (9)$$

Also , there is a useful relationships of the gamma function

$$\Gamma(n) = (n-1)!, \quad n \in N \qquad (10)$$

$$\Gamma(z) = \frac{\pi}{\sin(\pi z)\Gamma(1-z)} \qquad (11)$$

and from the last relation the researcher deduce that:

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \tag{12}$$

Definition of fractional derivatives:
In this paper, the researcher will identify two types of fractional derivative only

Definition:

- **Fractional derivative of Riemann-Liouville definition is: [8,4,9]**

  $$ {}_aD_t^{\alpha}\left[f(t)\right]=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{-n+\alpha+1}}\,dx \tag{13}$$

  Since n is positive integer number and $n-1<\alpha\le n$, f(t) is one time integrable .

  In the above definition if f(t)=c, c is constant, ${}_aD_t^{\alpha}\left[f(t)\right]\neq 0$

Definition:
Caputo fractional derivative: **[2, 4,10]**
Caputo developed the formula (13) which is not zero when f(t)=constant, so he defined Riemann-Liouville in another way as

$$ {}_aD_t^{\alpha} f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t \frac{f^{(n)}(x)}{(t-x)^{-n+\alpha+1}}\,dx, & n-1<\alpha<n\\[2ex] \dfrac{d^n f(t)}{dt^n}, & \alpha=n\end{cases} \tag{14}$$

The Mittag-Leffler Function: [5,13]
The Mittag-Leffler function of one-parameter is :

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{15}$$

Which was introduced by Mittag-Leffler and studied by Wiman.
The Mittag-Leffler function of two-parameter is:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},(\alpha>0,\ \beta>0) \tag{16}$$

And the result from the previous definition is :

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum\frac{z^k}{k!}=e^z \tag{17}$$

Furthermore, $E_{\alpha,\beta}^{(k)}(z)=\frac{d^k}{dy^k}E_{\alpha,\beta}(z)=\sum\frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+)}$ is the derivative of Mittag-Leffler function in two parameters.

Integral transforms:
Laplace Integral transform: [2,6,12]

$$L(f(t),s)=\int_0^{\infty} f(t)e^{-st}ds=\tilde{f}(s) \qquad , \mathrm{Re(s)}>0 \tag{18}$$

Where L is the Laplace operator .
The inverse Laplace transform :

$$f(t)=L^{-1}(\tilde{f}(s),t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\tilde{f}(s)ds,\ c=\mathrm{Re}(s) \tag{19}$$

Fourier-sine integral transform:[2,6,8]

$$F_e(f(x),\zeta)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x)f(x)dx \tag{20}$$

Solution of (FPDE) in xz plane :
Consider the time **(FPDE)** of Caputo fractional order as:

$$\frac{\partial^{\beta}u(x,z,t)}{\partial t^{\beta}}=\left[a\frac{\partial^2 u(x,z,t)}{\partial x^2}+b\frac{\partial^2 u(x,z,t)}{\partial z^2}\right], \quad n-1<\beta\le n, \tag{21}$$

With conditions as:

$$\frac{\partial^m u(x,z,0)}{\partial t^m}=b_m(x,z),\ m=0,1,2,...,n-1,\ for\ \ x,z>0 \tag{22}$$

$$u(x,0,t)=u(0,z,t)=1 \quad t>0 \tag{23}$$

$$u(x,z,t),\frac{\partial^m u(x,z,t)}{\partial x^m},\frac{\partial^m u(x,z,t)}{\partial z^m}\to 0\ ,\ m=0,1,2,...,n-1\ for\ \ x^2+z^2\to\infty \tag{24}$$

Use the Fourier-sine integral transform and conditions (23), (24). Then Equations (21) and (22) lead to

$$\frac{\partial^{\beta}U(\zeta,\xi,t)}{\partial t^{\beta}}=-(a\zeta^2+b\xi^2)U(\zeta,\xi,t)+\frac{2}{\pi}\zeta\xi \tag{25}$$

$$U^{(m)}(\zeta,\xi,0)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}\sin(\xi z)\sin(\zeta x)b_m(x,z)dxdz=b_m(\zeta,\xi) \tag{26}$$

Hence the Laplace transform of Eq. (25) is

$$p\tilde{U}(\zeta,\xi,p)+\zeta^2\tilde{U}(\zeta,\xi,p)+(a\zeta^2+b\xi^2)p^{\beta}\tilde{U}(\zeta,\xi,p) -\sum_{n=0}^{m}(a\zeta^2+b\xi^2)p^{\beta-n-1}U^{(n)}(\zeta,\xi,0)=\frac{2}{\pi}\frac{\zeta\xi}{p} \tag{27}$$

$$\tilde{U}(\zeta,\xi,p)=\frac{2\zeta\xi}{\pi p\left(p+(b\xi^2+a\zeta^2)p^{\beta}+(b\xi^2+a\zeta^2)\right)} +\sum_{m=0}^{n-1}\frac{(b\xi^2+a\zeta^2)b_m(\zeta,\xi)p^{\beta-m-1}}{p+(b\xi^2+a\zeta^2)p^{\beta}+(b\xi^2+a\zeta^2)} \tag{28}$$

The inverse Laplace transform **[7,13]** of Eq. (28) by using the relation

$$L^{-1}\left\{\frac{n!}{p^{\mu-\lambda}\left(p^{\lambda}\pm c\right)^{n+1}},t\right\}=t^{\lambda n+\mu-1}E^{(n)}_{\lambda,\mu}\left(\mp ct^{\lambda}\right),\quad \left(\operatorname{Re}(p)>|c|^{\frac{1}{\lambda}}\right) \tag{29}$$

Then Eq. (28) leads to

$$U(\zeta,\xi,t)=\frac{2}{\pi}\frac{1}{\zeta\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(b\xi^2+a\zeta^2)^{k+1}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}\left(-(b\xi^2+a\zeta^2)/t^{\beta-1}\right)+\sum_{m=0}^{n-1}b_m(\zeta,\xi) \times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(b\xi^2+a\zeta^2)^{(k+1)}t^{2-\beta}E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-(b\xi^2+a\zeta^2)/t^{\beta-1}\right) \tag{30}$$

So the inverse Fourier- sine integral transform of Eq. (30) is:

$$u(x,z,t)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}\sin(\xi z)\sin(\zeta x)\sin(\xi z)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{\beta k-1}\left[\frac{2}{\pi}\frac{1}{\zeta\,\xi}t^{\beta+1}E^{(k)}_{\beta-1,\beta+k+1}\left(-\frac{1}{t^{1-\beta}(a\zeta^2+b\xi^2)}\right) +\sum_{m=0}^{n-1}b_m(\zeta,\xi)t^{2m}E^{(k)}_{\beta-1,k+1}\left(-\frac{1}{t^{1-\beta}(a\zeta^2+b\xi^2)}\right)\right]d\zeta d\xi \tag{31}$$

which is the exact solution of (21).

Special Case:
The time fractional wave equation:
When b=0 and $0<\beta\le 1$, the special case of the equation (26) is the wave equation (see references **[8,10, 13],** for which its formula as:

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} = a\frac{\partial^{2} u(x,t)}{\partial x^{2}}, \quad 0<\beta\le 1 \tag{32}$$

$$u(x,0)=b_0(x), \;\; x,z>0 \tag{33}$$

$$u(0,t)=1, \quad t>0 \tag{34}$$

$$u(x,t), \frac{\partial u(x,t)}{\partial x}\to 0, \quad x\to\infty \tag{35}$$

Use the Fourier- sine integral transform and the Laplace transform respectively and conditions (34), (35), so Eq. (32) is:

$$\tilde{U}(\zeta,p)=\frac{2}{\pi}\frac{1}{\zeta}\sum_{k=0}^{\infty}(-1)^k\zeta^{2k+2}\frac{1}{p^{\beta k+\beta+1}\left(p^{1-\beta}+a\,\zeta^2\right)^{k+1}}+b_0(\zeta)\sum_{k=0}^{\infty}(-1)^k \times\zeta^{2k+2}\frac{1}{p^{\beta k+1}\left(p^{1-\beta}+a\,\zeta^2\right)^{k+1}} \tag{36}$$

The Laplace inverse transform and the Fourier-sine inverse integral transform respectively of Eq. (36) is:

$$u(x,t)=\frac{2}{\pi}\int_0^{\infty}\sin(\zeta\, x)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta^{2k+2}t\left\{\begin{array}{l}\frac{2}{\pi\zeta}t^k E^{(k)}_{1-\beta,2+\beta k}\left(-a^2\zeta^2t^{1-\beta}\right)\\ +ab_0(\zeta)t^{1-\beta}E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-a\zeta^2t^{1-\beta}\right)\end{array}\right\}d\zeta \tag{37}$$

which is the special case of the solution (31).

Solution of FPDE of The fractional Rayliegh-Stokes problem as special case:
Consider the FPDE as:

$$\frac{\partial^{\beta} u(x,z,t)}{\partial t^{\beta}}=\left(\nu+\alpha D_t^{\beta}\right)\left[\frac{\partial^2 u(x,z,t)}{\partial x^2}+\frac{\partial^2 u(x,z,t)}{\partial z^2}\right] \tag{38}$$

Since $u(x,z,t)$ = function in xz plane, t =time , ν, α = constants with respect to (x , z , t) and D_t^{β} = Caputo fractional derivative with n-1<β≤n.
The corresponding conditions of Eq.(38) are

$$\frac{\partial^m u(x,z,0)}{\partial t^m}=b_m(x,z), m=0,1,\dots,n-1, \quad x,\ z>0 \tag{39}$$

$$u(x,0,t)=u(0,z,t)=U, \quad for \quad t>0 \tag{40}$$

Moreover, the following condition has to be satisfied:

$$u\left(x,z,t\right),\frac{\partial^{m}u\left(x,z,t\right)}{\partial x^{m}},\frac{\partial^{m}u\left(x,z,t\right)}{\partial z^{m}}\to 0,\ m=0,1,2,...,n-1 \quad for \quad x^{2}+z^{2}\to\infty \tag{41}$$

Use the non-dimensional quantities:

$$u^{*}=\frac{u}{U},\ x^{*}=\frac{xU}{\nu},\ t^{*}=\frac{tU^{2}}{\nu},\ \eta=\alpha\frac{U^{2}}{\nu^{2}},\ z^{*}=\frac{zU}{\nu} \tag{42}$$

Equations (38 - 42) Reduce to non-dimension equations as follows (For reducing the dimensionless mark "*" is deleted here)

$$\frac{\partial^{\beta}u(x,z,t)}{\partial t^{\beta}}=\left(1+\eta D_{t}^{\beta}\right)\left[\frac{\partial^{2}u(x,z,t)}{\partial x^{2}}+\frac{\partial^{2}u(x,z,t)}{\partial z^{2}}\right],\quad n-1<\beta\leq n,\quad \eta=\frac{\alpha}{\nu} \tag{43}$$

$$\frac{\partial^{m}u(x,z,0)}{\partial t^{m}}=b_{m}(x,z),\ m=0,1,2,...,n-1,\ for\ x,z>0 \tag{44}$$

$$u\left(0,z,t\right)=1,u\left(z,0,t\right)=1 \quad t>0 \tag{45}$$

$$u\left(x,z,t\right),\frac{\partial^{m}u\left(x,z,t\right)}{\partial x^{m}},\frac{\partial^{m}u\left(x,z,t\right)}{\partial z^{m}}\to 0,\ m=0,1,2,...,n-1 \quad for \quad x^{2}+z^{2}\to\infty \tag{46}$$

Use the Fourier- sine integral transform and conditions (45), (46). Then Eqs. (43) and (44) lead to

$$\frac{\partial U(\zeta,\xi,t)}{\partial t}=-(\zeta^{2}+\xi^{2})\left(1+\eta D_{t}^{\beta}\right)U(\zeta,\xi,t)+\frac{2}{\pi}\zeta\xi \tag{47}$$

$$U^{(m)}(\zeta,\xi,0)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\sin(\xi z)\sin(\zeta x)b_{m}(x,z)dxdz=b_{m}(\zeta,\xi) \tag{48}$$

Hence the Laplace transform of Eq. (47) is

$$p\tilde{U}(\zeta,\xi,p)+\zeta^{2}\tilde{U}(\zeta,\xi,p)+\eta(\xi^{2}+\zeta^{2})p^{\beta}\tilde{U}(\zeta,\xi,p)-\sum_{m=0}^{n-1}\eta(\xi^{2}+\zeta^{2})p^{\beta-m-1}U^{(m)}(\zeta,\xi,0)=\frac{2}{\pi}\frac{\zeta\xi}{p} \tag{49}$$

$$\tilde{U}(\zeta,\xi,p)=\frac{2\zeta\xi}{\pi p\left(p+(\xi^{2}+\zeta^{2})\eta p^{\beta}+(\xi^{2}+\zeta^{2})\right)}+\sum_{m=0}^{n-1}\frac{(\xi^{2}+\zeta^{2})\eta b_{m}(\zeta,\xi)p^{\beta-m-1}}{p+(\xi^{2}+\zeta^{2})\eta p^{\beta}+(\xi^{2}+\zeta^{2})} \tag{50}$$

By using the following relation :

$$L^{-1}\left\{\frac{n!\,p^{\lambda-\mu}}{\left(p^{\lambda}\mp c\right)^{n+1}}\right\}=t^{\lambda n+\mu-1}E_{\lambda,\mu}^{(n)}\left(\pm ct^{\lambda}\right),\quad\left(\mathrm{Re}\,al(p)>|c|^{\frac{1}{\lambda}}\right) \tag{51}$$

So the inverse Laplace transform of Eq. (50) leads to:

$$U(\zeta,\xi,t)=\frac{2}{\pi}\frac{1}{\zeta\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\xi^2+\zeta^2)^{k+1}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}\left(-\eta(\xi^2+\zeta^2)t^{1-\beta}\right)+\sum_{m=0}^{n-1}\eta b_m(\zeta,\xi)$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\xi^2+\zeta^2)^{(k+1)}t^{2-\beta}E^{(k)}_{1-\beta,\beta k-\beta+2}\left(-\eta(\xi^2+\zeta^2)t^{1-\beta}\right) \tag{52}$$

where $E^{(k)}_{\alpha,\beta}(y)=\frac{d^k}{dy^k}E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{(j+k)!\,y^j}{j!\Gamma(\alpha j+\alpha k+)}$ is the Mittag-Leffler function in two parameters **[14]**.

So the inverse Fourier- sine integral transform of Eq. (52).

$$u(x,z,t)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty}\sin(\xi z)\,\sin(\zeta x)\sin(\xi z)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1}t^{\beta k-1}\left[\frac{2}{\pi}\frac{1}{\zeta\,\xi}t^{\beta+1}E^{(k)}_{\beta-1,\beta+k+1}\left(-\frac{1}{(\xi^2+\zeta^2+)\,\eta}t^{\beta-1}\right)\right.$$
$$\left.+\sum_{m=0}^{n-1}\eta\, b_m\,(\zeta\,,\,\xi)t^{\,2m}E^{(k)}_{\beta-1,k+1}\left(-\frac{1}{(\xi^2+\zeta^2)\,\eta}t^{\beta-1}\right)\right]d\zeta\,d\xi \tag{53}$$

which is the exact solution of (43).

Special Case:
Fractional Rayliegh-Stokes problem:
Now consider the following two cases of fractional Rayliegh-Stokes problem (see Fang and others (2006)):

Case (1): when $0<\beta\le 1$:
Then equations (43),(44),(45) and (46) lead to

$$\frac{\partial u(x,z,t)}{\partial t}=\left(1+\eta D_t^{\beta}\right)\left[\frac{\partial^2 u(x,z,t)}{\partial x^2}+\frac{\partial^2 u(x,z,t)}{\partial z^2}\right],\quad 0<\beta\le 1 \tag{54}$$

$$u\,(x,z,0)=b_0\,(x,z)\,,\quad x,z>0 \tag{55}$$

$$u\,(x,0,t)=u\,(0,z,t)=1,\quad t>0 \tag{56}$$

$$u\,(x,z,t),\,\frac{\partial u\,(x,z,t)}{\partial x},\frac{\partial u\,(x,z,t)}{\partial z}\to 0,\quad for\quad x^2+z^2\to\infty \tag{57}$$

Use the Fourier- sine integral transform and the Laplace transform respectively and conditions (56), (57). So Eq. (54) is:

$$\widetilde{U}\left(\zeta\ ,\xi\ ,p\right)=\frac{2}{\pi}\frac{1}{\zeta\ \xi}\sum_{k=0}^{\infty}(-1)^{k}(\xi^{2}+\zeta^{2})^{(k+1)}\frac{p^{-\beta\,k-\beta-1}}{\left(p^{1-\beta}+\eta\ (\xi^{2}+\zeta^{2})\right)^{k+1}}+\eta\ b_{0}(\zeta,\xi)\sum_{k=0}^{\infty}(-1)^{k}$$
$$\times(\xi^{2}+\zeta^{2})^{(k+1)}\frac{p^{-\beta\,k-1}}{\left(p^{1-\beta}+\eta\ \ (\xi^{2}+\zeta^{2})\right)^{k+1}} \tag{58}$$

Use the inverse Laplace transform and the inverse Fourier- sine integral transform respectively of Eq. (58) , then it leads to

$$u\ (x\ ,\ z\ ,t)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\sin(\xi\ z)\sin(\zeta\ x)\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k\ !}(\xi^{2}+\zeta^{2})^{(k+1)}$$
$$\times t\begin{Bmatrix}\frac{2}{\pi\ \zeta\ \xi}t^{k}E_{1-\beta,2+\beta\,k}^{(k)}\left(-\eta^{2}(\xi^{2}+\zeta^{2})t^{1-\beta}\right)\\+\eta\ b_{0}\ (\zeta,\xi)t^{1-\beta}E_{1-\beta,\beta\,k-\beta+2}^{(k)}\left(-\eta\ (\xi^{2}+\zeta^{2})t^{1-\beta}\right)\end{Bmatrix}d\zeta\ d\xi \tag{59}$$

So Eq. (59) is special case of equation (53).
Now take the special cases of case (1):
1.1 When $b_0\ (\zeta\ ,\ \xi)=0$, then Eq. (59) leads to:

$$u\left(x\ ,z\ ,t\right)=\frac{4}{\pi^{2}}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\sin\left(\xi\ z\right)\sin\left(\zeta\ x\right)}{\zeta\ \xi}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k\ !}(\xi^{2}+\zeta^{2})^{(k+1)}t^{k+1}E_{1-\beta,2+\beta\,k}^{(k)}\left(-\eta^{2}\ \zeta^{2}t^{1-\beta}\right)d\zeta\ d\xi \tag{60}$$

Eq. (60) is the result obtained by Fang and others **[6]**.
1.2 When $b_0\ (\zeta\ ,\ \xi)=0$, $\beta=1$, then Eq. (59) leads to:

$$u\ (x\ ,z\ ,t)=1-\frac{4}{\pi^{2}}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\sin\left(\zeta\ x\right)\sin\left(\xi\ z\right)}{\zeta\ \xi}e^{\left(\frac{-(\zeta^{2}+\xi^{2})}{1+\eta(\zeta^{2}+\xi^{2})}t\right)}d\zeta d\xi \tag{61}$$

So Eq. (61) is the result obtained by Fetacau and Corina **[2]**.

Case (2): When $1<\beta\leq2$:
Then equations (43-46) lead to:

$$\frac{\partial\ u(x\ ,,\ z,\ t)}{\partial\ t}=\left(1+\eta\ D_{t}^{\beta}\right)\left[\frac{\partial^{2}u\ (x\ ,z\ ,t)}{\partial\ x^{2}}+\frac{\partial^{2}u\ (x\ ,z\ ,t)}{\partial\ x^{2}}\right],\quad 1<\beta\leq2 \tag{62}$$

$$u\ \ (x\ ,z\ ,0)=b_{0}\ (x\ ,\ z)\ ,\ x\ ,z>0 \tag{63}$$

$$u_{\ t}\ (x\ ,z\ ,0)=b_{\ 1}(x\ ,\ z),\ x>0 \tag{64}$$

$$u\ (0\ ,z\ ,t)=u\ (x\ ,0\ ,t)=1,\quad t>0 \tag{65}$$

$$u(x,z,t),\ \frac{\partial\,u(x,z,t)}{\partial\,x},\frac{\partial u(x,z,t)}{\partial z}\rightarrow0\quad for\quad x^{2}+z^{2}\rightarrow\infty \tag{66}$$

Use the same strategies used in equations (43-46) ,
So the solution of Eq. (62) is:

$$u(x,t)=\frac{2}{\pi}\int_0^\infty\int_0^\infty \sin(\zeta x)\sin(\xi z)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1} t^{\beta k-1}\left[\frac{2}{\pi}\frac{1}{\zeta\,\xi}t^{\beta+1}E^{(k)}_{\beta-1,\beta+k+1}\left(-\frac{1}{(\zeta^2+\xi^2)\,\eta}t^{\beta-1}\right)\right.$$
$$\left.+\eta b_0(\zeta,\xi)E^{(k)}_{\beta-1,k}\left(-\frac{1}{(\zeta^2+\xi^2)\eta}t^{\beta-1}\right)+\eta b_1(\zeta,\xi)t^2E^{(k)}_{\beta-1,k+1}\left(-\frac{1}{(\zeta^2+\xi^2)\eta}t^{\beta-1}\right)\right]d\zeta d\xi \tag{67}$$

Now take the special cases of case 2:

2.1 When $\beta=2$, then Eq. (67) leads to:

$$u(x,z,t)=\frac{2}{\pi}\int_0^\infty\int_0^\infty \sin(\zeta x)\sin(\xi z)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1} t^{2k-1}\left[\frac{2}{\pi\,\zeta\,\xi}t^2E^{(k)}_{1,k+2}\left(-\frac{1}{(\zeta^2+\xi^2)\,\eta}t\right)\right.$$
$$\left.+\eta\, b_0(\zeta,\xi)E^{(k)}_{1,k}\left(-\frac{1}{(\zeta^2+\xi^2)\,\eta}t\right)+\eta\, b_1(\zeta,\xi)t^2E^{(k)}_{1,k+1}\left(-\frac{1}{(\zeta^2+\xi^2)\,\eta}t\right)\right]d\zeta\, d\xi \tag{68}$$

2.2 When $\beta=2$, $b_0(\zeta,\xi)=0$, $b_1(\zeta,\xi)=0$, then Eq. (67) leads:

$$u(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(\zeta x)\sin(\xi z)}{\zeta\,\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{\eta}\right)^{k+1} t^{2k+1}E^{(k)}_{1,k+2}\left(-\frac{1}{(\zeta^2+\xi^2)\,\eta}t\right)d\zeta\, d\xi \tag{69}$$

Eq. (69) is the result obtained by Salim and El-Kahlout **[12]**.

Now take the special cases of the last case (2.2) :

2.2.1 When $b_0(\zeta,\xi)=0$, then Eq. (69) leads to :

$$u(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(\zeta x)\sin(\xi z)}{\zeta\,\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\zeta^2+\xi^2)^{(k+1)}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}\left(-\eta^2\zeta^2t^{1-\beta}\right)d\zeta\, d\xi \tag{70}$$

Eq. (70) is the result obtained by Fang and others **[10]**.

2.2.2 When $b_0(\zeta,\xi)=0$, $\beta=1$, then Eq. (69) leads to :

$$u(x,z,t)=1-\frac{4}{\pi^2}\times\int_0^\infty\int_0^\infty \frac{\sin(\xi z)\sin(\zeta x)}{\zeta\,\xi}e^{\left(\frac{-(\zeta^2+\xi^2)}{1+\eta(\zeta^2+\xi^2)}t\right)}d\zeta d\xi \tag{71}$$

Eq. (71) is the result obtained by Fetacau and Corina **[2]**.

Conclusion

The exact solution of two types of fractional partial differential equations (FPDE) was obtained, where the fractional orders were of the caputo type. The solution strategy i :

First: find Fourier- sine integral transform of the FPDE, then find the Laplace integral transform of it.

Second: find the inverse of the Laplace transform of the integral of the equation, then find the inverse Fourier-sine integral transform.
With this strategy, the exact solution of the two equations was obtained, and some results were obtained from that. Researchers can use another fractional operators such as the Katjopla operator to obtain different results.

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