Generalized $n$-Closed Sets and Generalized $n$-Continuous Functions

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Abstract

The notion of a generalized $n$-closed set is introduced and the basic properties of these sets are established. A useful characterization of the generalized $n$-closed sets and a new property of the $n$-closure operator are proved. The concept of a generalized $n$-continuous function along with two related classes functions are developed.

Mathematics Subject Classification: 54C10, 54D10

Keywords: gn-open set, n-open set, gn-continuous function, n-continuous function, n-closure, n-interior.

1 Introduction

The concept of an $n$-open set was introduced in [1]. In this note we continue this line of investigation by introducing generalized $n$-closed (briefly, gn-closed) sets. A useful characterization of these sets is proved. Specifically we show that a set $A$ is gn-closed if and only if $n\text{Cl}(A) = A$. In general the gn-closed sets are better behaved and more useful than the n-open sets, although the sets do not necessarily form a minimal structure. Also a useful property of the n-closure operator is established. It is proved that for every subset $A$ of a topological space $X$ $n\text{Cl}(A) = A$ or $n\text{Cl}(A) = X$. The notion of a gn-continuous function is defined and the basic properties of these functions are
developed. Conditions equivalent to gn-continuity are established. Also two classes of related functions, gn-closed functions and gn-irresolute functions, are introduced.

2 Preliminaries

The symbols $X$ and $Y$ represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set $A$ are signified by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively.

**Definition 2.1** Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a minimal structure (briefly an $m$-structure) on $X$ \cite{2}, if $\emptyset \in m_X$ and $X \in m_X$.

**Definition 2.2** A subset $A$ of a space $X$ is said to be n-open \cite{1} if $\text{Int}(A) \neq \text{Cl}(A)$. A subset of $X$ is called n-closed if its complement is n-open.

**Theorem 2.3** \cite{1} A subset $A$ of a space $X$ is n-open if and only if $A$ is not clopen.

**Corollary 2.4** \cite{1} A subset $A$ of a space $X$ is n-open if and only if $X - A$ is n-open.

Thus the n-open sets coincide with the n-closed sets.

**Definition 2.5** Let $A$ be a subset of a space $X$. The n-interior of $A$ \cite{1} is denoted by $\text{nInt}(A)$ and given by $\text{nInt}(A) = \bigcup\{U \subseteq X : U \subseteq A \text{ and } U \text{ is n-open}\}$. The n-closure of $A$ \cite{1} is denoted by $\text{nCl}(A)$ and given by $\text{nCl}(A) = \bigcap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is n-closed}\}$.

**Theorem 2.6** \cite{1} The following statements hold for every set $A \subseteq X$:

(a) $\text{nInt}(X - A) = X - \text{nCl}(A)$.

(b) $\text{nCl}(X - A) = X - \text{nInt}(A)$.

(c) $x \in \text{nCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every n-open set $U$ containing $x$.

**Theorem 2.7** \cite{1} If $X$ is a space, then

(a) $\text{nCl}(X) = X$.

(b) $\text{nInt}(\emptyset) = \emptyset$. 
Theorem 2.8 [1] If $X$ is a discrete space, then

(a) $n\text{Int}(A) = \emptyset$ for every set $A \subseteq X$.

(b) $n\text{Cl}(A) = X$ for every set $A \subseteq X$.

Theorem 2.9 [1] If $U$ is an $n$-open set and $U = A \cup B$, then either $A$ is $n$-open or $B$ is $n$-open.

Definition 2.10 A function $f : X \rightarrow Y$ is said to be $n$-continuous [1] if $f^{-1}(V)$ is $n$-open in $X$ for every proper nonempty open set $V \subseteq Y$.

See [1] for additional properties and notation concerning $n$-open sets.

3 Generalized n-Closed Sets

Definition 3.1 A subset $A$ of a space $X$ is said to be generalized $n$-closed (briefly gn-closed) if whenever $A \subseteq U$ and $U$ is open, then $n\text{Cl}(A) \subseteq U$. A subset of $X$ is called generalized $n$-open (briefly gn-open) if its complement is gn-closed.

The collection of gn-closed sets may not form a minimal structure since it may not contain $\emptyset$.

Theorem 3.2 Let $A$ be a subset of a space $X$. Then $A$ is gn-open if and only if $F \subseteq n\text{Int}(A)$ whenever $F \subseteq A$ and $F$ is closed.

Example 3.3 Let $X = \{a,b,c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b,c\}\}$. The $n$-closed sets are $\{a\}, \{b\}, \{a,c\}$, and $\{b,c\}$. The gn-closed sets are $\{a\}, \{b\}, \{a,c\}, \{b,c\}, \{c\}, X$, and $\emptyset$. The set $\{c\}$ is gn-closed but not $n$-closed and the set $\{a,b\}$ is closed but not gn-closed.

Obviously $n$-closed sets are gn-closed. Example 3.3 shows that in general the two collections are not equal. Also from Example 3.3 the gn-closed sets and the gn-open sets do not coincide and the gn-closed sets are not in general closed under union. The fact that $n$-open sets are not closed under either union or intersection is illustrated by Example 3.3.

Example 3.4 Let $X = \{a,b,c\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$. All sets in $X$ are gn-closed.

Example 3.5 If $X$ is a discrete space, then $X$ is the only gn-closed set.
Example 3.6 Let $X$ denote the real numbers with the usual topology. Since there are no proper nonempty clopen sets, all proper nonempty sets are $n$-closed and hence all sets are $gn$-closed.

Theorem 3.7 Let $A$ be a subset of a space $X$. Then $A$ is $gn$-closed if and only if $nCl(A) = A$.

Proof. For the sufficiency assume that $A$ is $gn$-closed. If $A$ is open, then $nCl(A) \subseteq A$ and hence $nCl(A) = A$. If $A$ is not open, then $A$ is not clopen and hence $A$ is $n$-closed. Thus it follows from the definition of the $n$-closure operator that $nCl(A) = A$.

The necessity follows immediately from the definition.

Corollary 3.8 Let $A$ be a subset of a space $X$. Then $A$ is $gn$-open if and only if $nInt(A) = A$.

Theorem 3.9 The $gn$-closed sets in a space $X$ are closed under arbitrary intersection.

Proof. Let $A_\alpha$ be a $gn$-closed set for every $\alpha \in \mathcal{A}$. Then using Theorem 3.7 we obtain $nCl(\cap_{\alpha \in \mathcal{A}}A_\alpha) \subseteq \cap_{\alpha \in \mathcal{A}}nCl(A_\alpha) = \cap_{\alpha \in \mathcal{A}}A_\alpha$ and hence $nCl(\cap_{\alpha \in \mathcal{A}}A_\alpha) = \cap_{\alpha \in \mathcal{A}}A_\alpha$. Thus $\cap_{\alpha \in \mathcal{A}}A_\alpha$ is $gn$-closed.

Remark 3.10 If $X$ is discrete, then $X$ is the only $gn$-closed set in $X$ and hence in the above proof $A_\alpha = X$ for every $\alpha \in \mathcal{A}$.

Corollary 3.11 The $gn$-open sets in a space $X$ are closed under arbitrary union.

Theorem 3.12 Let $A$ be a subset of a space $X$. Then $nCl(A) = A$ or $nCl(A) = X$.

Proof. If $A$ is $n$-closed, then by the definition of $n$-closure $nCl(A) = A$. Assume $A$ is not $n$-closed. If there is no $n$-closed set that contains $A$, then $nCl(A) = X$. Assume $F$ is an $n$-closed set such that $A \subseteq F$. Then $F = A \cup (F - A)$. Since $F$ is $n$-closed and $A$ is not $n$-closed, it follows from Theorem 2.9 that $F - A$ is $n$-closed. (Recall that a set is $n$-closed if and only if it is $n$-open.) Since it’s complement $X - (F - A)$ is also $n$-closed and $A \subseteq F \cap (X - (F - A))$, it follows from the definition of the $n$-closure operator that $nCl(A) \subseteq F \cap (X - (F - A))$. Since $F \cap (X - (F - A)) = A$, it follows that $nCl(A) = A$.

Corollary 3.13 Let $A$ be a subset of a space $X$. Then $nCl(nCl(A)) = nCl(A)$. 

Corollary 3.14 Let $A$ be a subset of a space $X$. Then $n\text{Cl}(A)$ is $gn$-closed.

Corollary 3.15 If $A$ is a subset of a space $X$, then $A$ is $gn$-closed if and only if $A = n\text{Cl}(B)$ for some set $B \subseteq X$.

Corollary 3.16 If $A$ is a proper subset of a space $X$, then $A$ is $gn$-closed if and only if $n\text{Cl}(A) \neq X$.

Corollary 3.17 The collection of all $gn$-closed sets of a space $X$ is the set $\{A \subseteq X : n\text{Cl}(A) \neq X\} \cup \{X\}$.

Corollary 3.18 Let $A$ be a subset of a space $X$. If $A$ is proper, $gn$-closed, and not $n$-closed, then $A$ is the intersection of two $n$-closed sets.

4 Generalized n-Continuous Functions

Definition 4.1 A function $f : X \to Y$ is said to be generalized $n$-continuous (briefly $gn$-continuous) if $f^{-1}(F)$ is $gn$-closed in $X$ for every closed set $F \subseteq Y$.

Remark 4.2 If $X$ is a discrete space, then for every space $Y$ there is no $gn$-continuous function $f : X \to Y$. Note that $f^{-1}(\emptyset) = \emptyset$, which is not $gn$-closed in $X$.

Theorem 4.3 The following conditions are equivalent for a function $f : X \to Y$:

(a) $f$ is $gn$-continuous.

(b) $f^{-1}(V)$ is $gn$-open for every open set $V \subseteq Y$.

(c) $f^{-1}(\text{Int}(B)) \subseteq n\text{Int}(f^{-1}(B))$ for every set $B \subseteq Y$.

(d) $n\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B))$ for every set $B \subseteq Y$.

Proof. (a) $\Rightarrow$ (b) Let $V \subseteq Y$ be open. Then, using (a) and Theorem 3.7, we have $X - f^{-1}(V) = f^{-1}(Y - V) = n\text{Cl}(f^{-1}(Y - V)) = n\text{Cl}(X - f^{-1}(V)) = X - n\text{Int}(f^{-1}(V))$. Thus $f^{-1}(V) = n\text{Int}(f^{-1}(V))$ and by Corollary 3.8 $f^{-1}(V)$ is $gn$-open.

(b) $\Rightarrow$ (c) Let $B \subseteq Y$. By (b) $f^{-1}(\text{Int}(B))$ is $gn$-open. Hence by Corollary 3.8 $f^{-1}(\text{Int}(B)) = n\text{Int}(f^{-1}(\text{Int}(B))) \subseteq n\text{Int}(f^{-1}(B))$.

(c) $\Rightarrow$ (d) Let $B \subseteq Y$. Then $X - f^{-1}(\text{Cl}(B)) = f^{-1}(Y - \text{Cl}(B)) = f^{-1}(\text{Int}(Y - B)) \subseteq n\text{Int}(f^{-1}(Y - B)) = n\text{Int}(X - f^{-1}(B)) = X - n\text{Cl}(f^{-1}(B))$. 


Therefore \( \text{nCl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B)) \).

\( (d) \Rightarrow (a) \) Let \( F \subseteq Y \) be closed. It follows from \( (d) \) that \( \text{nCl}(f^{-1}(F)) \subseteq f^{-1}(\text{Cl}(F)) = f^{-1}(F) \). By Theorem 3.7 \( f^{-1}(F) \) is gn-closed and hence \( f \) is gn-continuous.

**Theorem 4.4** Assume \( X \) is not discrete. If \( f : X \to Y \) is \( n \)-continuous, then \( f \) is gn-continuous.

**Remark 4.5** If \( X \) is discrete and \( Y \) is indiscrete, then every function \( f : X \to Y \) is \( n \)-continuous but not gn-continuous.

**Example 4.6** Let \( X = \{a, b, c\} \) have the topologies \( \tau = \{X, \emptyset, \{a, b\}, \{c\}\} \) and \( \sigma = \{X, \emptyset, \{a, b\}\} \). The identity function \( f : (X, \tau) \to (X, \sigma) \) is gn-continuous but not \( n \)-continuous. Note that \( f^{-1}(\{c\}) \) is gn-closed but not \( n \)-closed.

**Definition 4.7** A function \( f : X \to Y \) is said to be generalized \( n \)-closed (briefly gn-closed) if \( f(F) \) is gn-closed in \( Y \) for every gn-closed set \( F \subseteq X \).

**Theorem 4.8** The following conditions are equivalent for a function \( f : X \to Y \):

(a) \( f \) is gn-closed.

(b) \( f(n\text{Cl}(A)) \) is gn-closed for every set \( A \subseteq X \).

(c) \( n\text{Cl}(f(A)) \subseteq f(n\text{Cl}(A)) \) for every set \( A \subseteq X \).

Proof. (a) \( \Rightarrow \) (b) By Corollary 3.14 \( n\text{Cl}(A) \) gn-closed for every set \( A \subseteq X \).

(b) \( \Rightarrow \) (c) Let \( A \subseteq X \). Then \( n\text{Cl}(f(A)) \subseteq n\text{Cl}(f(n\text{Cl}(A))) = f(n\text{Cl}(A)) \).

(c) \( \Rightarrow \) (a) \( A \subseteq X \) be gn-closed. Then using (c) we obtain \( n\text{Cl}(f(A)) \subseteq f(n\text{Cl}(A)) = f(A) \). Therefore \( f(A) = n\text{Cl}(f(A)) \) and hence \( f(A) \) is gn-closed and \( f \) is gn-closed.

**Definition 4.9** A function \( f : X \to Y \) is said to be generalized \( n \)-irresolute (briefly gn-irresolute) if \( f^{-1}(F) \) is gn-closed in \( X \) for every gn-closed set \( F \subseteq Y \).

The proof of the following theorem is analogous to that of Theorem 4.8.

**Theorem 4.10** The following conditions are equivalent for a function \( f : X \to Y \):

(a) \( f \) is gn-irresolute.

(b) \( f^{-1}(n\text{Cl}(A)) \) is gn-closed for every set \( A \subseteq Y \).

(c) \( n\text{Cl}(f^{-1}(A)) \subseteq f^{-1}(n\text{Cl}(A)) \) for every set \( A \subseteq Y \).
References


Received: September 21, 2021; Published: October 9, 2021