The \( k \)-th Largest Numbers of Maximum Independent Sets in Quasi-Forest Graphs

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Abstract

Let \( G = (V, E) \) be a simple undirected graph. A subset \( I \) of the vertex set \( V(G) \) is independent if there is no edge of \( G \) between any two vertices of \( I \). A maximum independent set is an independent set of maximum size. A graph \( G \) with vertex set \( V(G) \) is called a quasi-forest graph, if there exists a vertex \( x \in V(G) \) such that \( G - x \) is a forest. In this paper we complete the determination of the \( k \)-th (\( 3 \leq k \leq \lfloor n/2 \rfloor \)) largest numbers of maximum independent sets among all quasi-forest graphs of order \( n \geq 6 \) and characterize the extremal graphs.

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1 Introduction

Let \( G = (V, E) \) be a simple undirected graph. The vertex set of a graph \( G \) is referred to as \( V(G) \), its edge set as \( E(G) \). A subset \( I \subseteq V(G) \) is an independent set of \( G \) if no two vertices of \( I \) are adjacent in \( G \). An independent set \( I' \) of \( G \) is called maximum if \( G \) has no independent set \( I \) with \( |I'| < |I| \). The set of all maximum independent sets of a graph \( G \) is denoted by \( \text{XI}(G) \) and its cardinality by \( \text{xi}(G) \).
For notation and terminology in graphs we follow [1] in general. A graph is connected when there is a path between every pair of vertices. A triangle-free graph is a graph in which no three vertices form a triangle of edges. An acyclic graph, one not containing any cycles, is called a forest. A connected forest is called a tree. The problem of determining the largest number of maximum independent sets of a graph was studied for various classes of graphs, including general graphs, trees, forests, graphs with at most one cycle, graphs with at most \( r \) cycle, connected graphs and triangle-free graphs, see [3, 7]. Lin [5] investigated the second largest and the third largest cardinality of \( xi(G) \) among all trees and forests. Recently, Lin and Jou [6] investigated the \( k \)-th largest cardinality of \( xi(G) \) among all forests of order \( n \).

A connected graph (respectively, graph) \( G \) with vertex set \( V(G) \) is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex \( x \in V(G) \) such that \( G - x \) is a tree (respectively, forest). The problem of determining the largest and the second largest numbers of maximum independent sets among all quasi-tree graphs and quasi-forest graphs was solved by Lin [4].

The purpose of this paper is to determine the \( k \)-th \((3 \leq k \leq \lfloor n/2 \rfloor)\) largest numbers of maximum independent sets among all quasi-forest graphs of order \( n \geq 6 \). Extremal graphs achieving these values are also given.

2 Preliminary

In this section, we describe some notations and preliminary results. For a graph \( G = (V, E) \), the cardinality of \( V(G) \) is called the order, and it is denoted by \( |G| \). A maximal connected subgraph of \( G \) is called a component of \( G \). A component of odd (respectively, even) order is called an odd (respectively, even) component. For a set \( A \subseteq V(G) \), the deletion of \( A \) from \( G \) is the graph \( G - A \) obtained from \( G \) by removing all vertices in \( A \) and their incident edges.

Two graphs \( G_1 \) and \( G_2 \) are disjoint if \( V(G_1) \cap V(G_2) = \emptyset \). The union of two disjoint graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 \) with vertex set \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \). Let \( nG \) be the short notation for the union of \( n \) copies of disjoint graphs isomorphic to \( G \).

Denote by \( P_n \) a path with \( n \) vertices and \( C_n \) a cycle with \( n \) vertices. Throughout this paper, for simplicity, let \( r = \sqrt{2} \).

**Lemma 2.1.** ([2], [3]) If \( G \) is the union of two disjoint graphs \( G_1 \) and \( G_2 \), then \( xi(G) = xi(G_1) \cdot xi(G_2) \).

The result of the largest number of maximum independent sets among all trees is described in Theorem 2.2.
Theorem 2.2. ([2], [3]) If $T$ is a tree of order $n \geq 2$, then

$$x_i(T) \leq t_1(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even,} \\ r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $x_i(T) = t_1(n)$ if and only if $T = T_1(n)$, where

$$T_1(n) = \begin{cases} T_{1e}(n), & \text{if } n \text{ is even,} \\ T_{1o}(n), & \text{if } n \text{ is odd.} \end{cases}$$

The graphs $T_{1e}(n)$ and $T_{1o}(n)$ are shown in Figure 1.

![Figure 1: The graphs $T_{1e}(n)$ and $T_{1o}(n)$](image)

Define the graph $F_i(n)$ of order $n \geq 2$, $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ as follows.

$$F_i(n) = \begin{cases} T_{1e}(2i) \cup \frac{n-2i}{2} P_2, & \text{if } n \text{ is even,} \\ T_{1e}(2i) \cup P_1 \cup \frac{n-2i-1}{2} P_2, & \text{if } n \text{ is odd.} \end{cases}$$

Let $f_i(n) = x_i(F_i(n))$. For simple calculation, we have that

$$f_i(n) = \begin{cases} r^{n-2} + r^{n-2i}, & \text{if } n \text{ is even,} \\ r^{n-3} + r^{n-2i-1}, & \text{if } n \text{ is odd.} \end{cases}$$

The result of the $k$-th $(1 \leq k \leq \lfloor n/2 \rfloor)$ largest numbers of maximum independent sets among all forests is described in Theorem 2.3.

Theorem 2.3. ([2], [3], [6]) For integers $k$, $n \geq 2$ and $1 \leq k \leq \lfloor n/2 \rfloor$. If $F$ is a forest of order $n$ having $F \neq F_i(n)$, for $i = 1, 2, \ldots, k-1$, then $x_i(F) \leq f_k(n)$. Furthermore, $x_i(F) = f_k(n)$ if and only if $F = F_k(n)$ or $2T_{1e}(4) \cup F_1(n-8)$ with $k = 4$.

The result of the largest number of maximum independent sets among all quasi-tree graphs is described in Theorem 2.4.

Theorem 2.4. ([4]) If $Q$ is a quasi-tree graph of order $n \geq 2$, then

$$x_i(Q) \leq q_t(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even,} \\ r^{n-1} + 1, & \text{if } n \text{ is odd.} \end{cases}$$
Furthermore, $xi(Q) = qt_1(n)$ if and only if $Q = QT_1(n)$, where

$$QT_1(n) = \begin{cases} T_{1e}(n), & \text{if } n \text{ is even}, \\ QT_{1o}(n) \text{ or } C_5, & \text{if } n \text{ is odd}. \end{cases}$$

The graph $QT_{1o}(n)$ is shown in Figure 2.

The result of the second largest number of maximum independent sets among all quasi-tree graphs of even order is described in Theorem 2.5.

**Theorem 2.5.** ([4]) If $Q$ is a quasi-tree graph of even order $n \geq 6$ having $Q \neq QT_1(n)$, then $xi(Q) \leq r^{n-2}$ with the equality holding if and only if $Q = QT_{2e}(n)$ or $T_8$ or $P_6$, where $QT_{2e}(n)$, $T_8$ and $P_6$ are shown in Figure 3.

![Figure 2: The graph $QT_{1o}(n)$](image1)

The results of the largest and the second largest numbers of maximum independent sets among all quasi-forest graphs are described in Theorems 2.6 and 2.7, respectively.

**Theorem 2.6.** ([4]) If $Q$ is a quasi-forest graph of order $n \geq 2$, then

$$xi(Q) \leq qf_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even}, \\ 3r^{n-3}, & \text{if } n \text{ is odd}. \end{cases}$$

Furthermore, $xi(Q) = qf_1(n)$ if and only if $Q = QF_1(n)$, where

$$QF_1(n) = \begin{cases} F_1(n), & \text{if } n \text{ is even}, \\ C_3 \cup F_1(n-3), & \text{if } n \text{ is odd}. \end{cases}$$

![Figure 3: The graphs $QT_{2e}(n)$, $T_8$ and $P_6$](image2)
Theorem 2.7. ([4]) If $Q$ is a quasi-forest of order $n \geq 4$ having $Q \neq QF_1(n)$, then
\[
xi(Q) \leq qf_2(n) = \begin{cases} 
3r^{n-4}, & \text{if } n \text{ is even,} \\
5r^{n-5}, & \text{if } n \text{ is odd.}
\end{cases}
\]
Furthermore, $xi(Q) = qf_2(n)$ if and only if $Q = QF_2(n)$, where
\[
QF_2(n) = \begin{cases} 
F_2(n) \text{ or } C_3 \cup F_1(n-3), & \text{if } n \text{ is even}, \\
QT_{1o}(5) \cup F_1(n-5) \text{ or } C_5 \cup F_1(n-5), & \text{if } n \text{ is odd.}
\end{cases}
\]

3 Main results

In this section, we determine the $k$-th ($3 \leq k \leq \lfloor n/2 \rfloor$) largest values of $xi(G)$ among all quasi-forest graphs of order $n \geq 6$. Moreover, the extremal graphs achieving these values are also determined.

Define the graphs $QF_i(n)$ of order $n \geq 6$, $i = 3, 4, \ldots, \lfloor n/2 \rfloor$ as follows.
\[
QF_i(n) = \begin{cases} 
QT_{1o}(2i-1) \cup F_1(n-2i+1), & \text{if } n \text{ is even}, \\
QT_{1o}(2i+1) \cup F_1(n-2i-1), & \text{if } n \text{ is odd.}
\end{cases}
\]
Let $qf_i(n) = xi(QF_i(n))$. For simple calculation, we have that
\[
qf_i(n) = \begin{cases} 
r^{n-2} + r^{n-2i}, & \text{if } n \text{ is even,} \\
r^{n-1} + r^{n-2i-1}, & \text{if } n \text{ is odd.}
\end{cases}
\]

Lemma 3.1. If $Q$ is a quasi-forest graph of odd order $n \geq 7$ with $Q \neq QF_i(n)$ for $i = 1, 2, \ldots, k-1$ and $3 \leq k \leq (n-1)/2$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if $Q = QF_k(n)$ or $C_3 \cup F_2(n-3)$ for $k = 3$.

Proof. Since $f_1(n) < qf_k(n)$ for $n$ is odd, by Theorem 2.3, we assume that $Q$ is not a forest. Then there exists a component $H$ containing at least one cycle, where $|H| = m$. We consider the following two cases.

Case 1: $m$ is even. Since $H$ contains at least one cycle, it follows that $H \neq QT_1(m)$. By Lemma 2.1, Theorems 2.3 and 2.5, we have that
\[
xi(Q) = xi(H \cup (Q - V(H))) = xi(H) \cdot xi(Q - V(H)) \leq r^{m-2} \cdot r^{n-m-1} = r^{n-3} < r^{n-1} + r^{n-2k-1} = qf_k(n).
\]
Case 2: $m$ is odd. Since $H$ contains at least one cycle, it follows that $m \geq 3$. For the case that $Q - V(H) \neq F_1(n-m)$, by Lemma 2.1, Theorems 2.3 and 2.4, we have that

$$xi(Q) = xi(H \cup (Q - V(H)))$$
$$= xi(H) \cdot xi(Q - V(H))$$
$$\leq qt_1(m) \cdot f_2(n-m)$$
$$= (r^{m-1} + 1) \cdot 3^{n-m-4}$$
$$= 3^{n-5} + 3^{n-m-4}$$
$$\leq 3^{n-5} + 3^{n-7}$$
$$= 9^{n-7}$$
$$= qf_3(n).$$

Furthermore, the equalities holding imply that $m = k = 3$, $H = C_3$ and $Q - V(H) = F_2(n-3)$, that is, $Q = C_3 \cup F_2(n-3)$.

On the other hand, we assume that $Q - V(H) = F_1(n - m)$. Since $Q \neq QF_i(n)$ for $i = 1, 2, \ldots, k - 1$, by Lemma 2.1, Theorems 2.3 and 2.4, we have that

$$xi(Q) = xi(H \cup (Q - V(H)))$$
$$= xi(H) \cdot xi(Q - (V(H)))$$
$$\leq \left\{ \begin{array}{ll}
(qt_1(m) - 1) \cdot f_1(n - m), & \text{if } m \leq 2k - 1, \\
qt_1(m) \cdot f_1(n - m), & \text{if } m \geq 2k + 1,
\end{array} \right.$$
$$= \left\{ \begin{array}{ll}
r^{m-1} \cdot r^{n-m}, & \text{if } m \leq 2k - 1, \\
(r^{m-1} + 1) \cdot r^{n-m}, & \text{if } m \geq 2k + 1,
\end{array} \right.$$
$$= \left\{ \begin{array}{ll}
r^{n-1}, & \text{if } m \leq 2k - 1, \\
r^{n-1} + r^{n-2k-1}, & \text{if } m \geq 2k + 1,
\end{array} \right.$$
$$\leq r^{n-1} + r^{n-2k-1}$$
$$= qf_k(n).$$

Furthermore, the equalities holding imply that $m = 2k + 1$, $H = QT_{1o}(2k + 1)$ and $Q - V(H) = F_1(n - 2k - 1)$. In conclusion, $Q = QF_k(n) = QT_{1o}(2k + 1) \cup F_1(n - 2k - 1).$ \hfill \Box

**Lemma 3.2.** If $Q$ is a quasi-forest graph of even order $n \geq 6$ with $Q \neq QF_i(n)$ for $i = 1, 2, \ldots, k - 1$ and $3 \leq k \leq n/2$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if $Q = F_k(n)$ or $QF_k(n)$ or $C_5 \cup F_1(n - 5)$ for $k = 3$ or $2T_{1e}(4) \cup F_1(n - 8)$, $C_3 \cup F_2(n - 3)$ for $k = 4$.

**Proof.** Since $f_k(n) = qf_k(n)$ for $n$ is even, by Theorem 2.3, we assume that $Q$ is not a forest. We consider the following two cases.
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Case 1: All components of $Q$ are even. Let $H'$ be an even component containing at least one cycle, where $|H'| = m'$. Note that $H' \neq QT_1(m')$. By Lemma 2.1, Theorems 2.3 and 2.5, we have that

\[ xi(Q) = xi(H' \cup (Q - V(H'))) \]
\[ = xi(H') \cdot xi(Q - V(H')) \]
\[ \leq r^{m'-2} \cdot r^{n-m'} \]
\[ = r^{n-2} \]
\[ < r^{n-2} + r^{n-2k} \]
\[ = qf_k(n). \]

Case 2: There is an odd component $H''$ of $Q$, where $H''$ is a tree of order $m''$. Suppose that $m'' \geq 3$, by Lemma 2.1, Theorems 2.2 and 2.6, then

\[ xi(Q) = xi(H'' \cup (Q - V(H''))) \]
\[ = xi(H'') \cdot xi(Q - (V(H''))) \]
\[ \leq r^{m''-3} \cdot 3r^{n-m''-3} \]
\[ = 3r^{n-6} \]
\[ < qf_k(n). \]

For the case that $m'' = 1$, by Theorem 2.7 and Lemma 3.1, we have that $Q - V(H) = QT_{1,o}(2k-1) \cup F_1(n-2k)$ or $C_5 \cup F_1(n-6)$ for $k = 3$ or $C_3 \cup F_2(n-4)$ for $k = 4$. In conclusion, $Q = QF_k(n) = QT_{1,o}(2k-1) \cup F_1(n-2k+1)$ or $C_5 \cup F_1(n-5)$ for $k = 3$ or $C_3 \cup F_2(n-3)$ for $k = 4$. \qed

The result for the $k$-th ($3 \leq k \leq \lfloor n/2 \rfloor$) largest numbers of maximal independent sets among all quasi-forest graphs, now follow from Lemmas 3.1 and 3.2, and it is summarized in the following theorem.

**Theorem 3.3.** For integers $k$, $n \geq 6$ and $3 \leq k \leq \lfloor n/2 \rfloor$. If $Q$ is a quasi-forest graph of order $n$ with $Q \neq QF_i(n)$, for $i = 1, 2, \ldots, k-1$, then $xi(Q) \leq qf_k(n)$. Furthermore, the equality holds if and only if

1. $Q = QF_k(n)$,
2. $Q = C_3 \cup F_2(n-3)$ for $n$ is odd and $k = 3$,
3. $Q = F_k(n)$ for $n$ is even,
4. $Q = C_5 \cup F_1(n-5)$ for $n$ is even and $k = 3$,
5. $Q = 2T_{1,e}(4) \cup F_1(n-8)$ for $n$ is even and $k = 4$,
6. $Q = C_3 \cup F_2(n-3)$ for $n$ is even and $k = 4$. 

References


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