Boundary Values of Harmonic Functions 
and Modulus of Continuity (Part II)

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Abstract

We will present a proof of part (b) of our main theorem in a previous paper, which is Theorem 1.4 below. It is about a necessary and sufficient condition for a harmonic function on the open unit disk to be uniquely represented as the Poisson integral of a Riemann-integrable function on the unit circle in terms of modulus of continuity.

Mathematics Subject Classification: 30E25

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1 Introduction

Let $U$ denote the open unit disk, $\overline{U}$ the closed unit disk and $T$ the unit circle on the complex plane $\mathbb{C}$. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ the set of all positive integers. Let $Har(U)$ denote the set of all harmonic functions on $U$. The abbreviation a.e. means almost everywhere.

In [2], pp. 12-14, there is a result of K. G. Ivanov (Lemma 1.1, Lemma 1.2 and Theorem 1.3), which proved that if $f : [a, b] \to \mathbb{R}$ is bounded measurable, then $\omega(t; \cdot; f) : [a, b] \to \mathbb{R}$ is also bounded measurable with respect to Lebesgue measure $\sigma$, where $\omega(t; e^{i\alpha}; f) = \sup |f(e^{i\alpha_1}) - f(e^{i\alpha_2})|$ over all $\alpha_1, \alpha_2 \in [\theta - t/2, \theta + t/2]$. As a corollary, by taking the case $f(a) = f(b)$, we get Ivanov's result is also true for functions of the form $f : T \to \mathbb{R}$. In [1], we defined $\tau(t; f) = \|\omega(t; \cdot; f)\|_{L^1(T)}$ for $t \in [0, 2\pi]$. Next we will prove some useful facts.
Theorem 1.1 For a nonconstant \( u \in \text{Har}(U) \),

(a) if \( p \in [1, +\infty) \), \( \beta \in \mathbb{N} \), \( t \in (0, 2\pi) \) and \( r \in (0, 1) \), then \( \omega_\beta(t; u_r)_p > 0 \);

(b) if \( t \in (0, 2\pi) \), then \( \inf_{r \in (0,1)} (\tau(t; u_r)/\omega_1(t; u_r)_\infty) > 0 \).

Proof. For (a), suppose \( u \in \text{Har}(U) \) is not constant on \( U \). Assume there are \( p \in [1, +\infty) \), \( \beta \in \mathbb{N} \), \( t_0 \in (0, 2\pi) \) and \( r_1 \in (0, 1) \) such that \( \omega_\beta(t_0; u_{r_1})_p = 0 \). Define \( g : \overline{U} \to \mathbb{R} \) by \( g(z) = u(r_1z) \) for all \( z \in \overline{U} \). Then \( g \) is harmonic on \( U \) and continuous on \( \overline{U} \). Using the proof of Theorem 1.7(b) in [1], we showed for all \( t \in (0, 2\pi) \), \( \omega^\beta(t; g)_p = \omega^\beta(t; g|_U)_p \). However, \( \omega_\beta(t_0; g|_U)_p = \omega_\beta(t_0; u_{r_1})_p \). So \( \omega^{\beta}_1(t_0; g)_p = 0 \). By Theorem 1.6(b) in [1], we see \( \sup_{r \in (0,1)} \| g_r - g(0) \|_p = 0 \), i.e. \( g \) is constant on \( \overline{U} \). So \( u|_{\overline{B(0,r_1)}} \) is constant. Applying the Cauchy-Riemann equations, \( u \) is constant on \( U \), which is a contradiction.

For (b), suppose \( u \in \text{Har}(U) \) is not constant on \( U \). By (a), we have \( \omega_1(t; u_r) > 0 \) for all \( t \in (0, 2\pi) \), \( r \in (0, 1) \). Assume there is \( t_0 \in (0, 2\pi) \) such that \( \inf_{r \in (0,1)} (\tau(t; u_r)/\omega_1(t; u_r)_\infty) = 0 \). Since \( u \in \text{Har}(U) \), so \( u_r \in C(T) \) for every \( r \in (0, 1) \). By the corollary of Ivanov’s theorem, \( \omega(t; \cdot; u_r) \) is bounded measurable for all \( t \in [0, 2\pi] \) and \( \tau(t; u_r) = \| \omega(t; \cdot; u_r) \|_{L^1(T)} \) is well-defined. For all \( n \in \mathbb{N} \), there is \( r_n \in (0, 1) \) such that

\[
0 < \frac{\tau(t_0; u_{r_n})}{\omega_1(t_0; u_{r_n})_\infty} < \frac{1}{n}, \quad \text{then} \quad \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{\omega(t_0; e^{i\theta}; u_{r_n})}{\omega_1(t_0; u_{r_n})_\infty} d\theta = 0.
\]

Define \( \psi_n(\theta) = \omega(t_0; e^{i\theta}; u_{r_n})/\omega_1(t_0; u_{r_n})_\infty \) for all \( \theta \in [0, 2\pi] \). Then \( \{ \psi_n \} \) is a sequence of measurable function on \([0, 2\pi]\) that converges to 0 in measure. So there exists a subsequence \( \{ \psi_{n_k} \} \) converges to 0 a.e. on \([0, 2\pi]\) with respect to Lebesgue measure. Without loss of generality, we may assume \( \lim_{k \to \infty} \psi_{n_k}(0) = 0 \). Then there exists a partition \( P \in [0, 2\pi] \) such that

1. \( P = \{ 0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi \} \), and \( t_0/2 < \theta_{j+1} - \theta_j \) for \( 0 \leq j < n \),
2. \( [0, 2\pi] \subseteq \bigcup_{i=0}^{n} [\theta_i - \frac{t_0}{2}, \theta_i + \frac{t_0}{2}] \), then \( T \subseteq \bigcup_{j=0}^{n-1} \{ e^{i\omega} : \omega \in [\theta_j - \frac{t_0}{2}, \theta_j + \frac{t_0}{2}] \} \),
3. for \( 0 \leq j < n \), \( \lim_{k \to \infty} \omega(t_0; e^{i\theta_j}; u_{r_{n_k}})/\omega_1(t_0; u_{r_{n_k}})_\infty = 0 \) and
4. for \( 0 \leq i < n \), \( [\theta_i - t_0/2, \theta_i + t_0/2] \cap [\theta_{i+1} - t_0/2, \theta_{i+1} + t_0/2] \neq \emptyset \).

Take \( \varepsilon_0 = 1/16 \). For \( 0 \leq j < n \), there exists \( k_j \in \mathbb{N} \) such that for all \( k \geq k_j \), \( \omega(t_0; e^{i\theta}; u_{r_{n_k}})/\omega_1(t_0; u_{r_{n_k}})_\infty < 1/16 \). Let \( k^* \) be the maximum of \( k_0, k_1, \ldots, k_{n-1} \). For all \( \theta \in \mathbb{R} \) and \( h \in [0, t_0] \), denote \( S_j \) to mean \( \{ e^{i\omega} : \omega \in [\theta_j - t_0/2, \theta_j + t_0/2] \} \) for some \( j \in \{ 0, 1, \ldots, n - 1 \} \). Then \( e^{i\theta}, e^{i(\theta+h)} \in S_j \cup S_{j+1} \) for \( 0 \leq j < n \). Since \( [\theta_j - t_0/2, \theta_j + t_0/2] \cap [\theta_{j+1} - t_0/2, \theta_{j+1} + t_0/2] \neq \emptyset \) by (4).
Boundary values of harmonic functions

Let $w_1$ be in this set. Then

$$|u_{r_{nk}^*}(e^{i\theta}) - u_{r_{nk}^*}(e^{i(\theta+h)})|$$

$$\leq |u_{r_{nk}^*}(e^{i\theta}) - u_{r_{nk}^*}(e^{i\omega_1})| + |u_{r_{nk}^*}(e^{i\omega_1}) - u_{r_{nk}^*}(e^{i(\theta+h)})|$$

$$< (1/16)\omega_1(t_0; u_{r_{nk}^*})_\infty + (1/16)\omega_1(t_0; u_{r_{nk}^*})_\infty = (1/8)\omega_1(t_0; u_{r_{nk}^*})_\infty.$$ 

For $e^{i\theta} \in S_f$, $|u_{r_{nk}^*}(e^{i\theta}) - u_{r_{nk}^*}(e^{i\omega_1})| \leq \omega(t_0; e^{i\theta}); u_{r_{nk}^*} < (1/16)\omega_1(t_0; u_{r_{nk}^*})_\infty.$

As $\theta \in \mathbb{R}$, $h \in [0, t_0]$ are arbitrary, we see $\omega_1(t_0; u_{r_{nk}^*})_\infty \leq (1/8)\omega_1(t_0; u_{r_{nk}^*})_\infty$. So $\omega_1(t_0; u_{r_{nk}^*})_\infty = 0$, which is a contradiction.

**Definition 1.2** For a function $g : T \rightarrow \mathbb{R}$, we say $g$ is measure-continuous at $z \in T$ if for every $\varepsilon > 0$, there exists an open arc $I$ centered at $z$ such that $\sigma(I \cap \{w \in T : |g(w) - g(z)| \geq \varepsilon\}) = 0$. Also, we say $g$ is measure-continuous a.e. on $T$ if the measure of the set of the points at which $g$ fails to be measure-continuous is zero.

**Theorem 1.3** For $f^* : T \rightarrow \mathbb{R}$, $f^* \in L^\infty(T)$ is measure-continuous a.e. on $T$ if and only if there exists $g : T \rightarrow \mathbb{R}$ such that (1) $f^* = g$ a.e. on $T$, (2) $g$ is bounded on $T$, (3) $g$ is continuous a.e. on $T$.

**Proof.** Suppose for $f^* : T \rightarrow \mathbb{R}$, there exists $g : T \rightarrow \mathbb{R}$ such that $f^* = g$ a.e. [\sigma] on $T$, $g$ is bounded on $T$ and $g$ is continuous a.e. on $T$.

We claim $g$ is measurable. This is due to $g$ is bounded and continuous a.e. on $T$, hence Riemann integrable on $T$. Then $f^* = g$ a.e. [\sigma] on $T$ implies $f^*$ is measurable and $f^* \in L^\infty(T)$. To see $f^*$ is measure-continuous a.e. on $T$, let $e^{i\theta} \in C(g) \cap E$, where $C(g)$ is the set of all points of continuity of $g$ and $E = \{e^{i\phi} : f^*(e^{i\phi}) = g(e^{i\phi})\}$. Let $\varepsilon_0 > 0$. Then “for all $w \in I, |g(w) - g(e^{i\theta})| < \varepsilon_0$” is true for some open arc $I$ centered at $e^{i\theta}$. So for all $w \in I \cap E$, $|f^*(w) - f^*(e^{i\theta})| < \varepsilon_0$. Then $\sigma(I \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0\}) = 0$. Since $\sigma(T \setminus C(g) \cap E)) = 0$, $f^*$ is measure-continuous a.e. on $T$.

Conversely, suppose $f^* \in L^\infty(T)$ is measure-continuous a.e. on $T$. Let $L_{f^*} = \{e^{i\theta} : e^{i\theta}$ is a Lebesgue point of $f^*\}$ and $m_c(f^*) = \{e^{i\theta} : e^{i\theta}$ is a measure-continuous point of $f^*\}$. Then $\sigma(T \setminus (L_{f^*} \cap m_c(f^*))) = 0$. For every $e^{i\theta} \in T$ and $n \in \mathbb{N}$, let $I_n(e^{i\theta})$ be the open arc centered at $e^{i\theta}$ with arc length $2\pi/n$. Then for every $e^{i\theta} \in T$ and $n \in \mathbb{N}$, the sequence $\{L_n(e^{i\theta})\} = \left\{\int_{I_n(e^{i\theta})} f^* d\sigma/\sigma(I_n(e^{i\theta}))\right\}$ is bounded as $|L_n(e^{i\theta})| \leq \|f^*\|_\infty < +\infty$. So $\{L_n(e^{i\theta})\}$ has a convergent subsequence, say $\{L_{nk}(e^{i\theta})\}$ with limit $L(e^{i\theta})$. Then $|L(e^{i\theta})| \leq \|f^*\|_\infty$.

For an open arc $J$ in $T$, define $\mu(f^*, J) = (\int_J f^* d\sigma)/\sigma(J)$. We claim that for every $e^{i\theta} \in m_c(f^*)$ and $\varepsilon > 0$, there exists an open arc $I$ centered at $e^{i\theta}$ such that for any open arc $I' \subseteq I$ (with the center of $I'$ may not be $e^{i\theta}$), we
have $|f^*(e^{i\theta}) - \mu(f^*, I')| < \varepsilon$. (The proof is as follows. Let $e^{i\theta} \in m_{c}(f^*)$ and $\varepsilon_0 > 0$. By definition of measure-continuous at $e^{i\theta}$, there exist an open arc $I$ centered at $e^{i\theta}$ such that $\sigma(I \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0/2\}) = 0$. Let $A = I' \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| < \varepsilon_0/2\}$ and $B = I' \cap \{w \in T : |f^*(w) - f^*(e^{i\theta})| \geq \varepsilon_0/2\}$. Then $\sigma(A) = \sigma(I')$ and $\sigma(B) = 0$.

For every open arc $I' \subseteq I$, we have

$$|\mu(f^*, I') - f^*(e^{i\theta})| \leq \left( \int_{I'} |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w) \right) / \sigma(I')$$

$$= \left( \int_{A} |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w) \right) / \sigma(I') + \left( \int_{B} |f^*(w) - f^*(e^{i\theta})| \, d\sigma(w) \right) / \sigma(I')$$

$$\leq (\varepsilon_0/2)(\sigma(A)/\sigma(I')) + 0 < \varepsilon_0.$$

Next we claim for every $e^{i\theta} \in m_{c}(f^*) \cap L_{f^*}$, $L : T \to \mathbb{R}$ is continuous at $e^{i\theta}$. To see that, let $e^{i\theta} \in m_{c}(f^*) \cap L_{f^*}$. We have

$$L(e^{i\theta}) = \lim_{k \to \infty} \frac{\int_{I_{n_k}(e^{i\theta})} f^*(w) \, d\sigma(w)}{\sigma(I_{n_k}(e^{i\theta}))} = \lim_{k \to \infty} \mu(f^*, I_{n_k}(e^{i\theta})) = f^*(e^{i\theta}).$$

Let $\varepsilon_0 > 0$. By the previous claim, there exists an open arc $I$ centered at $e^{i\theta}$ such that for every open arc $I' \subseteq I$ (the center of $I'$ may not be $e^{i\theta}$), $|\mu(f^*; I') - L(e^{i\theta})| < \varepsilon_0/2$.

Let $w_0 \in I$. Since $I$ is open, “for all $k \geq k_1, I_{n_k}(w_0) \subseteq I$” is true for some $k_1 \in \mathbb{N}$. Moreover, “for all $k \geq k_2, |L(w_0) - \mu(f^*, I_{n_k}, (w_0))| < \varepsilon_0/2$” is true for some $k_2 \in \mathbb{N}$. Take $k^* = \max\{k_1, k_2\}$. We have

$$|L(e^{i\theta}) - L(w_0)| \leq |L(e^{i\theta}) - \mu(f^*, I_{n_k}, (w_0))| + |\mu(f^*, I_{n_k}, (w_0)) - L(w_0)|$$

$$< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0.$$

So there exists $L : T \to \mathbb{R}$ such that $L = f^*$ a.e. $[\sigma]$ on $T$, $L$ is bounded on $T$ and $L$ is continuous a.e. on $T$.

In terms of modulus of continuity, we will prove our main theorem below, which gives a necessary and sufficient condition for a harmonic function on $U$ to be the Poisson integral of a Riemann integrable function on $T$.

**Theorem 1.4** For $u \in Har(U)$, $\lim_{t \to 0^+} \tau^*(t; u) = 0$ if and only if there is a unique $f^* \in R(T)$ such that $u = P[f^*]$, where $R(T)$ is the set of all Riemann integrable functions on $T$. (Recall by Lebesgue’s theorem, for every bounded function $f^* : T \to \mathbb{R}$, $f^*$ is Riemann integrable if and only if $f^*$ is continuous a.e. on $T$.)
Proof. Suppose \( u \in Har(U) \) satisfies \( \lim_{t \to 0^+} \tau^*(t; u) = 0 \). We claim \( u \) is bounded on \( U \) as follows. Since \( \lim_{t \to 0^+} \tau^*(t; u) = 0 \), if \( \epsilon_0 = 1 \), then for all \( t \in (0, \delta_0) \), \( \tau^*(t; u) < \epsilon_0 \) is true for some \( \delta_0 \in (0, 2\pi) \). So \( \limsup_{r \to 1^-} \tau(t; u) = \tau^*(t_0; u) < \epsilon_0 = 1 \), where \( t_0 = \delta_0/2 \).

By Theorem 1.1(b), let \( k_0 = \inf_{r \in (0,1)} \tau(t_0; u)/\omega_1(t_0, u) > 0 \). For every \( r(0, 1) \), \( \omega_1(t_0, u)_\infty \leq \tau(t_0, u)/k_0 \) and so

\[
\limsup_{r \to 1^-} \omega_1(t_0; u)_\infty = \inf_{0<\delta<1} \sup_{1-\delta<r<1} \omega_1(t_0; u)_\infty 
\leq (\inf_{0<\delta<1} \sup_{1-\delta<r<1} \tau(t_0; u))/k_0 = \limsup_{r \to 1^-} \tau(t_0; u)/k_0 < 1/k_0.
\]

By Theorem 1.6(b) in [1], \( \sup_{r \in (0,1)} \|u - u(0)\|_\infty \leq (4\pi/t)\omega_1(t_0; u)_\infty < 4\pi/(t_0k_0) < \infty \). So \( u \) is bounded on \( U \) and there is a unique \( g \in L^\infty(T) \) with \( u = P[g] \).

Next, we claim \( g \) is measure-continuous a.e. on \( T \) as follows. Suppose \( g \) is not measure-continuous at \( z \in T \). For every \( j \in N \), define \( K_j = \{ z \in T : \text{for all open arc } I \text{ centered at } z, 0 < \sigma(I \cap \{ w \in T : |g(w) - g(z)| \geq 1/j \}) \} \). Let \( r_n = 1 - 1/n \) for \( n = 1, 2, 3, \ldots \), then \( \lim_{n \to \infty} r_n = 1 \).

We claim for all \( j \in N \), \( K_j \cap \{ e^{i\theta} \in T : e^{i\theta} \text{ is a Lebesgue point of } g \} \subseteq \cap_{t \in (0, 2\pi)} \{ z \in T : \liminf_{t \to \infty} \omega(t; z, u_{rn}) \geq 1/j \}. \) The reason is as follows. Let \( j \in N \), \( z_0 \in K_j \cap \{ e^{i\theta} \in T : e^{i\theta} \text{ is a Lebesgue point of } g \} \) and \( t_2 \in (0, 2\pi) \). Let \( I_{z_0} = \{ e^{i\theta} \in T : \theta_0 - t_2/2 < \phi < \theta_0 + t_2/2 \} \), where \( z_0 = e^{i\theta_0} \) for some \( \theta_0 \in R \). Then \( 0 < \sigma(I_{z_0} \cap \{ w \in T : |g(w) - g(z_0)| \geq 1/j \}) \) and \( 0 < \sigma(I_{z_0} \cap \{ w \in T : |g(w) - g(z_0)| \geq 1/j \}) \cap \{ w \in T : w \text{ is a Lebesgue point of } g \} \). So there exists \( w_0 \in T \) such that

(1) \( w_0 \in I_{z_0} \cap \{ w \in T : |g(w) - g(z_0)| \geq 1/j \} \) and

(2) \( w_0 \) is a Lebesgue point of \( g \) (that means \( u \) has non-tangential limit at \( w_0 \)).

Since \( z_0 \) is also a Lebesgue point of \( g \), \( \lim_{n \to \infty} |u_{rn}(z_0) - u_{rn}(w_0)| = |g(z_0) - g(w_0)| \geq 1/j \) by (1). So for all \( n \in N \),

\[
|u_{rn}(z_0) - u_{rn}(w_0)| \leq \sup_{\alpha_1, \alpha_2 \in [\theta_0-t_2/2, \theta_0+t_2/2]} |u_{rn}(e^{i\alpha_1}) - u_{rn}(e^{i\alpha_2})| = \omega(t_2; z_0; u_{rn})
\]

and

\[
\frac{1}{j} \leq \lim_{n \to \infty} |u_{rn}(z_0) - u_{rn}(w_0)| = \liminf_{n \to \infty} |u_{rn}(z_0) - u_{rn}(w_0)|
= \sup_{n \geq 1} \inf_{k \geq n} |u_{rk}(z_0) - u_{rk}(w_0)| \leq \sup_{n \geq 1} \inf_{k \geq n} \omega(t_2; z_0; u_{rk}) = \liminf_{n \to \infty} \omega(t_2; z_0; u_{rn}).
\]

Next, we claim that for all \( j \in N \), \( \sigma(K_j) = 0 \). To see that let \( \epsilon_0 > 0 \) and \( j \) be a positive integer. Then "for every \( t \in (0, \delta_1) \), \( \tau^*(t; u) < \epsilon_0 \)" is true for
some $\delta_1 \in (0, 2\pi)$. Take $t_3 = \delta_1/2$. By Fatou's lemma,

$$\int_T \liminf_{n \to \infty} \omega(t_3; z; u_{r_n}) d\sigma(z) \leq \liminf_{n \to \infty} \int_T \omega(t_3; z; u_{r_n}) d\sigma(z) = \liminf_{n \to \infty} \tau(t_3; u_{r_n}) \leq \limsup_{r \to 1^-} \tau(t_3; u_r).$$

So $\int_T \liminf_{n \to \infty} \omega(t_3; z; u_{r_n}) d\sigma(z) \leq \tau^*(t_3; u) < \varepsilon_0$. Then

$$\frac{1}{j} \sigma(\{z \in T : \liminf_{n \to \infty} \omega(t_3; z; u_{r_n}) \geq \frac{1}{j}\}) \leq \int_T \liminf_{n \to \infty} \omega(t_3; z; u_{r_n}) d\sigma(z) \leq \tau^*(t_3; u) < \varepsilon_0.$$

By (a), $\sigma(K_j) = \sigma(K_j \cap \{e^{i\theta} : e^{i\theta} \text{ is a Lebesgue point of } g\}) \leq \sigma(\{z \in T : \liminf_{n \to \infty} \omega(t_3; z; u_{r_n}) \geq 1/j\}) < j\varepsilon_0$. Since $\varepsilon_0$ is arbitrary, for all $j \in \mathbb{N}$, $\sigma(K_j) = 0$, i.e. $g$ is measure-continuous a.e. on $T$. By Theorem 1.3, there exists $f^* \in R(T)$ such that $f^* = g$ a.e. $[\sigma]$ on $T$. So there is a unique $f^* \in R(T)$ such that $u = P[f^*]$, where the uniqueness is in the equivalence class sense of $L^\infty(T)$.

Conversely, suppose $u \in Har(U)$ and there is a unique $f^* \in R(T)$ with $u = P[f^*]$. Then $f^*$ is bounded on $T$. Let $M > 0$ be such that for all $z \in T, |f^*(z)| \leq M$. Since for all $z \in U, u(z) = P[f^*](z)$, so for all $z \in U, |u(z)| \leq M$. Now $f^*$ is Riemann integrable on $T$. So $f^*$ is continuous a.e. on $T$.

Define $\tilde{f}^* : [0, 2\pi] \to \mathbb{R}$ by $\tilde{f}^*(\theta) = f^*(e^{i\theta})$. Then $\tilde{f}$ is continuous a.e. on $[0, 2\pi]$. Let $C(\tilde{f}^*) = \{\theta \in [0, 2\pi] : \tilde{f}^* \text{ is continuous at } \theta\}$. Fix $\varepsilon_0 > 0$. For every $\theta \in (0, 2\pi) \cap C(\tilde{f}^*)$, there exists $\delta_\theta > 0$ such that for all $\alpha \in [0, 2\pi] \cap (\theta - \delta_\theta, \theta + \delta_\theta), |f^*(e^{i\alpha}) - f^*(e^{i\theta})| < \varepsilon_0/48$. Now the collection

$$\mathcal{I} = \{(\theta - \delta, \theta + \delta) : \theta \in (0, 2\pi) \cap C(\tilde{f}^*), (\theta - \delta, \theta + \delta) \subseteq (0, 2\pi), \delta \in (0, \delta_\theta)\}.$$

is a Vitali covering of the set $(0, 2\pi) \cap C(\tilde{f}^*) = 2\pi$. By the Vitali covering lemma, there is a finite pairwise-disjoint collection $\{I_1, I_2, \ldots, I_N\}$ of intervals in $\mathcal{I}$ such that $m((0, 2\pi) \cap C(\tilde{f}^*)) \cup \bigcup_{i=1}^N I_i < \pi\varepsilon_0/(8M)$. So for every $j = 1, 2, \ldots, N$, $I_j = (\theta_j - \delta_j, \theta_j + \delta_j) \subseteq (0, 2\pi)$ for some $\theta_j \in (0, 2\pi) \cap C(\tilde{f}^*)$, $\delta_j \in (0, \delta - \theta_j)$. Then there exists a positive number $\Delta$ such that $(2\Delta)N < \pi\varepsilon_0/(8M)$ and $\delta < \min\{\delta_1, \delta_2, \ldots, \delta_N, \pi\}$. For $j = 1, 2, \ldots, N$, let $I'_j = [\theta_j - \delta_j + \Delta, \theta_j + \delta_j - \Delta] \subseteq I_j$. Now $\delta_j - \Delta > 0$. So $m((0, 2\pi) \setminus \bigcup_{i=1}^N I'_i) < (\pi\varepsilon_0)/(8M) + (\pi\varepsilon_0)/(8M) = (\pi\varepsilon_0)/(4M)$.

Recall the Poisson kernel $P_r(\theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)$ for all $\theta \in \mathbb{R}$, $r \in [0, 1)$. Since for all $\delta \in (0, \pi], \lim_{r \to 1^-} P_r(\theta) = 0$ uniformly in $\theta \in [\delta, 2\pi - \delta]$, so $\lim_{r \to 1^-} P_r(\theta) = 0$ uniformly in $\theta \in [\Delta, 2\pi - \Delta]$. Then “for every $r \in [r^*, 1)$
and \( \theta \in [\Delta, 2\pi - \Delta] \), \(|P_r(\theta)| < (\varepsilon_0/(2M))(1/48)\) is true for some \( r^* \in (0, 1) \). Let \( j = 1, 2, \ldots, N \), \( r \in [r^*, 1) \), \( \theta \in [\theta_j - \delta_j + \Delta, \theta_j + \delta_j - \Delta] = I'_j \). We have

\[
|P[f^*](re^{i\theta}) - f^*(e^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)f^*(e^{it})dt - \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)f^*(e^{it})dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)(f^*(e^{it}) - f^*(e^{i\theta}))dt
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \left| P_r(\theta - t)f^*(e^{it}) - f^*(e^{i\theta}) \right| dt
\]

\[
\leq \left( \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)dt(2M) \right) + \left( \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)dt \right) \frac{\varepsilon_0}{48}
\]

\[
\leq \frac{1}{2\pi} \cdot \varepsilon_0 + \frac{1}{2\pi} \cdot \frac{1}{48}(2\pi - \theta_j - \delta_j) \cdot 2M
\]

\[
< \varepsilon_0/48 + \varepsilon_0/48 = \varepsilon_0/16
\]

due to \( 0 < \theta_j - \delta_j < 2\pi \) and \( 0 < 2\pi - \theta_j - \delta_j < 2\pi \). So for \( j = 1, 2, \ldots, N \), \( \theta \in I'_j, r_1, r_2 \in [r^*, 1) \), we have

\[
|P[f^*](r_1e^{i\theta}) - P[f^*](r_2e^{i\theta})| < |P[f^*](r_1e^{i\theta}) - f^*(e^{i\theta})| + |f^*(e^{i\theta}) - P[f^*](r_2e^{i\theta})|
\]

\[
< \varepsilon_0/16 + \varepsilon_0/16 = \varepsilon_0/8.
\]

Since \( u_{r^*} \) as a restriction of \( u \) to \( \partial \Omega(0, r^*) \) is continuous on \( T \), so for all \( t \in [0, 2\pi], z \in T \), we have \( \omega(t; z; u_{r^*}) \leq \omega(t; u_{r^*}) \). For every \( t \in [0, 2\pi] \), we have \( \tau(t; u_{r^*}) = ||\omega(t; z; u_{r^*})||_1(1 \leq \omega(t; u_{r^*}) \) and \( \lim_{t \to 0^+} \tau(t; u_{r^*}) = 0 \) by the uniform continuity of \( u_{r^*} \). So “for all \( t \in (0, t_1) \), \( \tau(t; u_{r^*}) < \varepsilon_0/4 \)” is true for some \( t_1 \in (0, 2\pi) \). Take \( t_2 = \min\{t_1, \varepsilon_0\pi/(4MN), \delta_1 - \Delta, \delta_2 - \Delta, \ldots, \delta_N - \Delta\} \), which is in \((0, 2\pi)\). Define \( I''_j = [\theta_j - \delta_j + \Delta + t_2/2, \theta_j + \delta_j - \Delta - t_2/2] \) for \( j = 1, 2, \ldots, N \). Since \( t_2 \leq \delta_j - \Delta \), we have \( t_2/2 < \delta_j - \Delta \). Now for every \( j = 1, 2, \ldots, N, \theta \in I''_j, r \in [r^*, 1) \), \( \alpha_1, \alpha_2 \in [\theta - t_2/2, \theta + t_2/2] \),

\[
|u_{r^*}(e^{i\alpha_1}) - u_{r^*}(e^{i\alpha_2})| \leq |u_{r^*}(e^{i\alpha_1}) - u_{r^*}(e^{i\alpha_2})| + |u_{r^*}(e^{i\alpha_1}) - u_{r^*}(e^{i\alpha_2})| + |u_{r^*}(e^{i\alpha_2}) - u_{r^*}(e^{i\alpha_2})|
\]

\[
< \varepsilon_0/8 + \omega(t_2; e^{i\theta}; u_{r^*}) + \varepsilon_0/8.
\]

So \( \omega(t_2; e^{i\theta}; u_{r^*}) < \varepsilon_0/4 + \omega(t_2; e^{i\theta}; u_{r^*}) \). Now for \( r \in [r^*, 1) \),

\[
\tau(t_2; u_{r^*}) = \int_T \omega(t_2; z; u_{r^*})d\sigma(z) = \frac{1}{2\pi} \int_0^{2\pi} \omega(t_2; e^{i\theta}; u_{r^*})d\theta
\]
\[
\frac{1}{2\pi} \int_{[0,2\pi] \setminus \bigcup_{j=1}^{N} I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta + \frac{1}{2\pi} \int_{\bigcup_{j=1}^{N} I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta \\
\leq \frac{1}{2\pi} (2M) \int \frac{\pi \varepsilon_0}{4M} + \sum_{j=1}^{N} \frac{1}{2\pi} \int_{I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta + \sum_{j=1}^{N} \frac{1}{2\pi} \int_{I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta \\
\leq \varepsilon_0 + \sum_{j=1}^{N} \frac{1}{2\pi} (2M) t_2 + \sum_{j=1}^{N} \frac{1}{2\pi} \int_{I_{j}'} \left( \frac{\varepsilon_0}{4} + \omega(t_2; e^{i\theta}; u_r) \right) d\theta \\
\leq \varepsilon_0 + \sum_{j=1}^{N} \frac{1}{2\pi} \int_{I_{j}'} \frac{\varepsilon_0}{4} \cdot \frac{\varepsilon_0}{4} + \sum_{j=1}^{N} \frac{1}{8\pi} \int_{I_{j}'} d\theta + \sum_{j=1}^{N} \frac{1}{2\pi} \int_{I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta \\
= \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{1}{2\pi} \int_{\bigcup_{j=1}^{N} I_{j}'} \omega(t_2; e^{i\theta}; u_r) d\theta \\
\leq \varepsilon_0 + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \omega(t_2; e^{i\theta}; u_r) d\theta \\
< \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \varepsilon_0.
\]

So for all \( t \in (0, t_2) \), we see \( \tau^*(t; u) = \lim sup_{\tau \to t^{-}} \tau(t; u_r) \leq \lim sup_{\tau \to t^{-}} \tau(t_2; u_r) = \inf_{0<\delta<1} \sup_{1-\delta<\tau<1} \tau(t_2; u_r) \leq \varepsilon_0 \). Therefore, \( \lim_{t \to 0^+} \tau^*(t; u) = 0 \).

References


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