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The Absolute Value Operation as an Effective Tool for Enhancing Students' Cognitive Activities in Calculus Courses

Meirav Amram

Shamoon College for Engineering, Ashdod, Israel

Miriam Dagan

Shamoon College for Engineering, Beer-Sheva, Israel

Pavel Satianov

Shamoon College for Engineering, Beer-Sheva, Israel

Michael Yoshpe

Shamoon College for Engineering, Ashdod, Israel

Abstract

The absolute value operation is, by definition, very simple, but a composition of this operation with other functions, and even with a linear function, is difficult to the students. The reason is that this work involves many technical steps. In this paper we give a shortcut to graphical solutions of such problems. We also offer the absolute value as an attractive tool to obtain different symmetries on graphs of equations.

Mathematics Subject Classification: 97D40, 97D50, 97D70

Keywords: absolute value, calculus teaching, challenging problems, visual representation

1 Introduction

The absolute value is a basic mathematical operation. It has some equivalent definitions, with the following being the most widespread: the absolute value $|x|$ of a real number x is the non-negative value of x (without regard to its sign); this means, $|x| = x$ for a positive x, $|x| = -x$ for a negative x, and $|0| = 0$. Sierpinska-Bobos-Pruncut [1] summarize five different definitions for the absolute value that show the diversity insight on this operation. Brumfiel [2] shows five similar definitions for the absolute value; he then considers one inequality that involves absolute values and solves it beautifully using the different definitions. The various definitions of the absolute value have led Wilhelmi-Godino-Lacasta to study the didactic effectiveness of those definitions [3]. Arcidiacono [4] says that in order to solve a problem related to the absolute value, students need to analyze it by breaking it down to cases; he therefore offers the graphic approach, which is very helpful in the classroom.

To demonstrate the problem behind learning this subject, we turn to the paper written by Chiarugi-Fracassina-Furinghetti [5]. The authors explain that when a student attempts to solve a problem concerning the absolute value, difficulties appear precisely when passing from the arithmetic part to the algebraic part of the solution. While the arithmetic part - checking the absolute value of numbers - can be quite clear, moving to the algebraic part - writing variables instead of numbers - causes difficulty for the student, who has trouble interpreting variables. In this paper we offer a graphical approach that enables us to shorten algebraic calculations.

We suggest an effective approach to graphical solutions of equations that involve compositions of linear functions and the absolute value operation. In the examples we a give a shortcut to a graphical solution of such problems (Section 2). Then we show the use of symmetries, that are derived by the absolute values operation, in order to understand graphs related to some equations with an absolute value (Section 3). We present the mathematical problems in the paper as examples from the classroom, and merge discussions between a lecturer and students, about the way they reach their solutions.

2 A composition of the absolute value operation with linear functions

In this section we show some examples of a composition of the absolute value operation with linear functions. We explain the shortcut of the investigation in

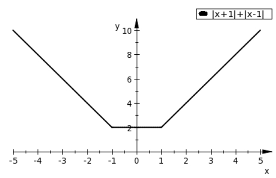

Figure 1: The function $|x+1|+|x-1|$

order to get the graph. This idea of composition produces a work with graphs that are broken lines. For the detailed construction of the broken line, see [6].

The lecturer starts with the following function: $f(x) = |x+1| + |x-1|$. By definition of the absolute value, he obtains

$$f(x) = \begin{cases} -2x, & x < -1; \\ 2, & -1 \leq x \leq 1; \\ 2x, & x > 1, \end{cases} \tag{1}$$

and the graph of this function (Figure 1) is shown on the board:

The lecturer next asks the students to sketch the graph of the function $f(x) = 2x - 3 - |x| - 2|x-4| + 3|x-5|$. After some minutes the students say that it is cumbersome, because of the tedious division to cases. So the lecturer presents a common idea that enables the students to sketch such graph almost instantly and to understand that it is a concatenation of a finite number of segments, given their endpoints. Let

$$f(x) = a_0 x + b + \sum_{i=1}^{n} a_i |x - x_i|$$

be a function, such that $x_1 < x_2 < \ldots < x_n$, $a_i \in R$ for $i = 0, 1, \ldots, n$, $a_i \neq 0$ for $i = 1, 2, \ldots, n$. Then, the function $f(x)$ is linear in the intervals $(-\infty, x_1), \ldots, (x_i, x_{i+1}), \ldots, (x_n, \infty)$ as a concatenation of linear functions, so the graph of $f(x)$ is a broken line with x_i as its vertices.

Now the work in the classroom is carried out by a discussion; the students give the vertices $0, 4, 5$ of the function $f(x) = 2x - 3 - |x| - 2|x-4| + 3|x-5|$ and they choose two other points outside the finite segments: -1 and 6. The function's values at these points are easily calculated, and by connecting these vertices they derive the graph of $f(x)$ as a broken line, as seen in Figure 2.

For an advanced use of this idea, we suggest investigating the graph of equations with two variables that are composed of absolute value and linear operations, such as $|x| + |y| + |x + y - 2| = 6$. The corresponding graph is described in Figure 3.

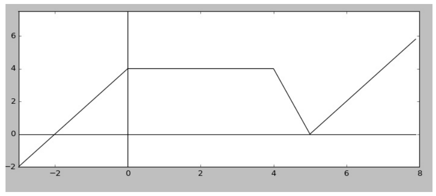

Figure 2: The broken line $2x - 3 - |x| - 2|x - 4| + 3|x - 5|$

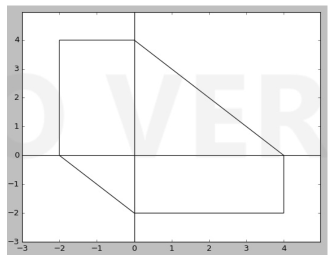

Figure 3: The graph of $|x| + |y| + |x + y - 2| = 6$

3 The absolute value and symmetries of graphs of equations with two and three variables

In this section we discuss the symmetry properties of the absolute value operation as a tool to obtain symmetries of graphs of equations in two and three variables.

We start with a discussion in the classroom. The lecturer presents the equation $|x| - |y| = 1$ and asks the students to solve it.

Student A uses the definition of the absolute value operation, and splits the investigation to 4 cases: for the first quadrant he has $x > 0, y > 0$, and he gets the equation $x - y = 1$; for the second quadrant he has $x < 0, y > 0$, and he gets the equation $-x - y = 1$; for the third quadrant he has $x < 0, y < 0$, and he gets the equation $-x + y = 1$; and for the fourth quadrant, where $x > 0, y < 0$, he gets the equation $x - y = 1$. He then obtains the graph of the equation $|x| - |y| = 1$ as a union of the 4 subgraphs.

Student B starts with the equation $x-y = 1$, whose graph is a straight line. She then adds an absolute value on x only as follows: $|x| - y = 1$. the graphs of this equation and the original one are the same for $x > 0$, and for $x < 0$ the equation gets the form of the reflection of the line $y = x - 1$ through the y-axis. At the next step, she looks at $|x| - |y| = 1$; by the same consideration - the part of the graph that is above the x-axis stays at place, while the negative part is replaced by the positives' reflections through the x-axis. The steps of her solution and the final graph appear in Figure 4.

Now the lecturer gives the instant solution, based on the general idea: the

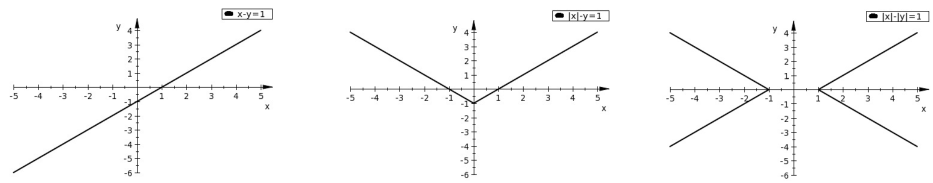

Figure 4: The steps for solving the equation $|x| - |y| = 1$

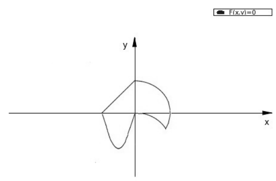

Figure 5: The assumed graph for the equation $F(x, y) = 0$

graph of an equation $F(|x|, |y|) = 0$ is symmetric in relation to the x-axis, the y-axis, and the origin of the coordinate system. If a point (a, b) satisfies this equation, then the points $(a, -b), (-a, b)$, and $(-a, -b)$ also satisfy the equation. So the graph of the equation $|x| - |y| = 1$ in the first quadrant looks like the part of the line $x - y = 1$. We can obtain the whole graph by the above-written symmetries.

The lecturer draws a graph of the equation $F(x, y) = 0$ on the board (Figure 5) and asks the students to supply the graphs of the equations $F(|x|, y) = 0$, $F(-|x|, y) = 0$, $F(x, |y|) = 0$, and $F(x, -|y|) = 0$.

There is a discussion in the classroom and then the students, with the guidance of the lecturer, conclude the following: When $x > 0$ the equation $F(|x|, y) = 0$ is the same as $F(x, y) = 0$, so the graph is the same as the original graph. According to the symmetry principal, the graph is symmetric with relation to the y-axis (left side of Figure 6). Similarly, if we want to solve the equation $F(-|x|, y) = 0$, we can see that for $x < 0$, the graph is the same as the original graph, and again, by the symmetry principal, the right part of the graph is a reflection of the left part (right side of Figure 6).

Similar considerations give the resulting graph for the solution of the equations $F(x, |y|) = 0$ and $F(x, -|y|) = 0$, which can be found in Figure 7.

Note that the process above can be also done in reverse: that is, given two graphs, where one contains a reflection of the other, we can find which sort of reflection was applied, and on which variables the absolute value was added. For instance, given the graph as shown in Figure 5 and the graph in the left side of Figure 6, we see that the right part in the first is reflected through the y-axis, and the result is the latter graph. We therefore conclude that the absolute value was applied on the first variable.

The lecturer continues and gives an example of equation $|x| + |y| + |z| = 1$.

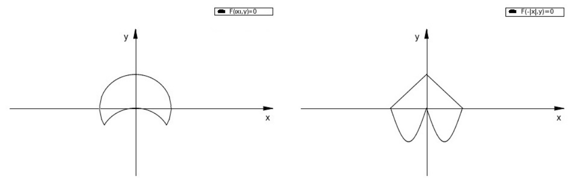

Figure 6: Graphs of the solutions for $F(|x|, y) = 0$ and $F(-|x|, y) = 0$

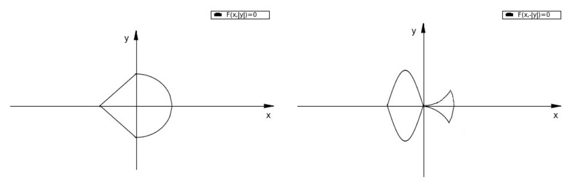

Figure 7: Graphs of the solutions for $F(x, |y|) = 0$ and $F(x, -|y|) = 0$

He discusses with the student how to get the related graph. The lecturer explains that because there are 3 variables here, the graph will be a figure in a 3-dimensional coordinate system. He then asks the students to describe the suitable figure. Student C says that there are 8 octants in the 3-dimensional coordinate system and begins to divide his work into 8 linear equations, according to the signs of the variables. Student D says that according to the previous example, it is enough to understand the graph in the first octant $(x > 0, y > 0, z > 0)$. In this octant the original equation becomes $x+y+z = 1$, and gives us a plane triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. According to the idea of the symmetry, the full figure is obtained by reflections of this triangle, in relation to the x-axis, the y-axis, and the origin of the coordinate system. The lecturer draws the figure on the board and asks the students if anyone knows the name of it, and only one of the students gives the right answer–it is a hollow Octahedron with 8 faces.

Conclusion. Our experience shows that the above-described approach to graphical solutions of equations with absolute value makes it possible for students to solve such problems without algebraic difficulties. Moreover, by this approach, students can understand the effectiveness of common mathematical ideas, such as linearity and continuity of the talked-about functions. Our approach also gives an opportunity to solve more complicated problems and increase cognitive interests of the students in the learning process. We also recommend the use of a scientific calculator, for example, an *fx-991ES PLUS*; the CALC operation of the calculator can enable prompt calculations of the values of functions, and the graphical programs of the calculator can be used for verification of results.

References

[1] A. Sierpinska, G. Bobos, A. Pruncut, Teaching absolute value inequalities to mature students, *Educational studies in mathematics,* **78** (2015), no. 3, 275 - 305. https://doi.org/10.1007/s10649-011-9325-2

[2] C. Brumfiel, Teaching the absolute value function, *Mathematics Teacher,* **73** (1980), no. 1, 24 - 30.

[3] M.R. Wilhelmi, J.D. Godino, E. Lacasta, Didactic effectiveness of mathematical definitions, The case of the absolute value, *International Electronic Journal of Mathematics Education,* **2** (2007), no. 2, 72 - 90.

[4] M.J. Arcidiacono, A visual approach to absolute value, *The Mathematics Teacher,* **76** (1983), no. 3, 197 - 201.

[5] I. Chiarugi, G. Fracassina, F. Furinghetti, Learning difficulties behind the notion of absolute value, *Proceedings of the annual Conference of the International Group for the Psychology of Mathematics Education,* **3** (1990), 231 - 238.

[6] P. Satianov, M. Fried, A. Amit, Broken-Line Functions with Unbroken Domains, *Mathematics Teacher,* **92** (1999), no. 7, 574 - 577.

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