Characterization of the Distance-$k$ Independent Dominating Sets of the $n$-Path

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Abstract

The distance between two vertices $u$ and $v$ in a graph $G$ equals the length of a shortest path from $u$ to $v$. A set $I$ of vertices is distance-$k$ independent if every vertex in $I$ is at distance at least $k+1$ to any other vertex of $I$. A set $I$ of vertices is distance-$k$ dominating if every vertex not belonging to $I$ is at distance at most $k$ of a vertex in $I$. A set of vertices is a distance-$k$ independent dominating set if and only if this set is a distance-$k$ independent set and a distance-$k$ dominating set. Note that in general counting the number of independent dominating sets in a graph is NP-complete [2]. In this paper, we want to characterize all the distance-$k$ independent dominating sets of the path $P_n$. Besides, we calculate the number of the distance-$k$ independent dominating sets of $P_n$.

Mathematics Subject Classification: 05C69

Keywords: dominating set, independent dominating set, distance-$k$ independent dominating set, path

1 Introduction

One of the fastest growing areas within graph theory is the study of domination and related subset problems. The theory of independent domination was proposed by Berge [1] and Ore [4] in 1962. The importance of an independent dominating set in the context of clustering wireless networks has been
widely acknowledged. The independent domination number of a graph is the minimum size of an independent dominating set of vertices. The independent dominating set (IDS) problem asks for the independent domination number in a graph. Gary and Johnson [2] showed that its NP-complete for general graphs. Goddard and Henning [3] offered a survey of selected recent results on independent domination in graphs.

Although there are many papers studying the independent dominating set (IDS) problem, we prefer to consider the number of the distance-k independent dominating sets in a graph. For a graph $G$, the set of all the distance-$k$ independent dominating sets of a graph $G$ is denoted by $\mathcal{I}_k(G)$ and its cardinality by $i_k(G)$. Denote $P_n$ a path of order $n$. In this paper, we want to characterize the sets $\mathcal{I}_k(P_n)$ and calculate the numbers $i_k(P_n)$, where $k \geq 1$ and $P_n$ is the $n$-path.

2 Characterization

In this section, we provide a constructive characterization of $\mathcal{I}_k(P_n)$, where $n \geq 1$ and $V(P_n) = \{1, 2, \ldots, n\}$. Besides, we calculate the number $i_k(P_n)$. Besides, we calculate the number of the distance-$k$ independent dominating sets of $P_n$. In order to give a constructive characterization of $\mathcal{I}_k(P_n)$, we introduce the following sets.

For $k \geq 1$ and $1 \leq i \leq k + 1$, let $\mathcal{T}_1, \mathcal{T}_2 = \ldots, \mathcal{T}_{2k+1}$ be the following sets.

\begin{align*}
\mathcal{T}_1 &= \{\{1\}\}, \\
\mathcal{T}_2 &= \{\{1\}, \{2\}\}, \\
\vdots \\
\mathcal{T}_i &= \{\{1\}, \ldots, \{i\}\}, \\
\vdots \\
\mathcal{T}_{k+1} &= \{\{1\}, \{2\}, \ldots, \{k+1\}\}, \\
\mathcal{T}_{k+2} &= \{\{1, k+2\}, \{2\}, \ldots, \{k+1\}\}, \\
\vdots \\
\mathcal{T}_{k+i} &= \{\{1, k+i\}, \{2, k+i\}, \ldots, \{i-1, k+i\}, \\
&\quad \{1, k+i-1\}, \{2, k+i-1\}, \ldots, \{i-2, k+i-1\}, \\
&\quad \ldots \\
&\quad \{1, k+2\}, \\
&\quad \{i\}, \{i+1\}, \ldots, \{k+1\}\}.
\end{align*}
Characterization of the distance-$k$ independent dominating sets of the $n$-path

$$
\mathcal{T}_{2k+1} = \{\{1, 2k+1\}, \{2, 2k+1\}, \ldots, \{k, 2k+1\}, \\
\{1, 2k\}, \{2, 2k\}, \ldots, \{k-1, 2k\}, \\
\vdots \\
\{1, k+2\}, \\
\{k+1\}\}.
$$

**Observation 1.** For $1 \leq n \leq 2k+1$, $\mathcal{I}_k(P_n) = \mathcal{T}_n$

**Observation 2.** For $1 \leq n \leq 2k+1$,

$$
|\mathcal{T}_n| = \begin{cases} 
  n, & \text{if } 1 \leq n \leq k+1, \\
  k+1 + \frac{s(s+1)}{2}, & \text{if } k+2 \leq n = (k+2) + s \leq 2k+1.
\end{cases}
$$

Let $\mathcal{A}$ be a collection of sets. We define that $\mathcal{A} \oplus a = \{A \cup \{a\} : A \in \mathcal{D}\}$.

For $n \geq 2k+2$, let

$$
\mathcal{T}_n = \bigcup_{i=1}^{k+1} \mathcal{T}_{n-k-i} \oplus \{n-i+1\}.
$$

We want to show that $\mathcal{I}_k(P_n) = \mathcal{T}_n$. First, we prove the following lemma.

**Lemma 2.1.** For $k \geq 1$ and $n \geq 2k+2$, $\mathcal{T}_n \subseteq \mathcal{I}_k(P_n)$.

**Proof.** We prove this lemma by induction on $n$, where $n \geq 2k+2$. For $n = 2k+2$, then

$$
\mathcal{T}_{2k+2} = \bigcup_{i=1}^{k+1} \mathcal{T}_{(2k+2)-k-i} \oplus \{(2k+2) - i + 1\}
$$

$$
= \{\{1, 2k+2\}, \{2, 2k+2\}, \ldots, \{k+1, 2k+2\}, \\
\{1, 2k+1\}, \{2, 2k+1\}, \ldots, \{k, 2k+1\}, \\
\vdots \\
\{1, k+2\}\}.
$$

Every set in $\mathcal{T}_{2k+2}$ is a distance-$k$ independent dominating set of $P_n$, so it's true for $n = 2k+2$. Assume that it's true for all $n' < n$ and let $I \in \mathcal{T}_n$, where $n \geq 2k+3$. Suppose $a$ is the largest number in $I$, then $a = n - i + 1$ for $i = 1, \ldots, k+1$. Let $I' = I - \{a\}$. Since $d(a, n-k-i) = (n-i+1) - (n-k-i) = k+1$, this means that $I' \in \mathcal{T}_{n-k-i}$ for $i = 1, \ldots, k+1$. By the induction hypothesis, $I' \in \mathcal{I}(P_{n-k-i})$. Note that $d(a, n-k-i) = k+1$, this means that $I$ is a distance-$k$ independent set of $P_n$. Now we want to show that $I$ is a distance-$k$ dominating set of $P_n$. Let $1 \leq j \leq n$ and $j \notin I$. We consider three cases.

**Case 1.** $1 \leq j \leq n-k-i$. Note that $I' \in \mathcal{I}_k(P_{n-k-i})$, then $j$ is at distance at most $k$ to some vertex in $I'$. 

Case 2. \( n-k-i+1 \leq j \leq a-1 \). Note that \( d(a, j) \leq (n-i+1) - (n-k-i+1) = k \), then \( j \) is at distance at most \( k \) to the vertex \( a \).

Case 3. \( a+1 \leq j \leq n \). Note that \( d(a, j) \leq n - (a+1) = n - (n-i+1+1) = i - 2 < k \), then \( j \) is at distance at most \( k \) to the vertex \( a \).

By Case 1, Case 2 and Case 3, we can see that \( j \) is at distance at most \( k \) to some vertex in \( I \), where \( I = I' \cup \{a\} \). Thus \( I \) is a distance-\( k \) dominating set of \( P_n \). Hence \( I \in \mathcal{T}_k(P_n) \), we complete the proof.

In the following theorem, we will show that \( \mathcal{T}_n \) is the characterization of \( \mathcal{S}_k(P_n) \).

**Theorem 2.2.** For \( n \geq 2k+2 \), \( \mathcal{S}_k(P_n) = \mathcal{T}_n \).

**Proof.** By Lemma 2.1, we obtain that \( \mathcal{T}_n \subseteq \mathcal{S}_k(P_n) \). Now we want to show that \( \mathcal{S}_k(P_n) \subseteq \mathcal{T}_n \) and prove it by induction on \( n \), where \( n \geq 2k+2 \). We can see that

\[
\mathcal{S}_k(P_{2k+2}) = \{1, 2k+2\}, \{2, 2k+2\}, \ldots, \{k+1, 2k+2\},
\]

\[
\{1, 2k+1\}, \{2, 2k+1\}, \ldots, \{k, 2k+1\},
\]

\[
\vdots
\]

\[
\{1, k+2\}\]

\[
= \bigcup_{i=1}^{k+1} \mathcal{T}_{(2k+2)-k-i} \oplus \{(2k+2) - i + 1\}
\]

\[
= \mathcal{T}_{2k+2}.
\]

So it’s true for \( n = 2k+2 \). Assume that it’s true for all \( n' < n \), where \( n \geq 2k+3 \). Suppose \( I \subseteq \mathcal{S}_k(P_n) \) and \( a \) is the largest number in \( I \). Thus \( a \) is at distance at most \( k \) to the vertex \( n \), say \( a = n-i+1 \) for some \( i \in \{1, \ldots, k+1\} \). Let \( I' = I - \{a\} \). Note that \( a - (k+1) = (n-i+1) - (k+1) = n-k-i \), then \( I' \subseteq \{1, 2, \ldots, n-k-i\} \). Since \( I \) is a distance-\( k \) independent set of \( P_n \), we can see that \( I' \) is a distance-\( k \) independent set of \( P_{n-k-i} \). Since \( I \) is a distance-\( k \) dominating set of \( P_n \) and \( d(a, j) \geq k+1 \) for \( j \in \{1, 2, \ldots, n-k-i\} \), we have that \( I' \) is a distance-\( k \) dominating set of \( P_{n-k-i} \). Hence \( I' \in \mathcal{S}_k(P_{n-k-i}) \), by the induction hypothesis, \( I' \in \mathcal{T}_{n-k-i} \). This means that \( I = I' \cup \{n-i+1\} \in \mathcal{T}_n \). So it’s true for \( n \) and \( \mathcal{S}_k(P_n) \subseteq \mathcal{T}_n \). We complete the proof.

For \( k \geq 1 \) and \( n \geq 2k+2 \), we provide a constructive characterization of \( \mathcal{S}_k(P_n) \), where \( \mathcal{T}_1, \mathcal{T}_1, \ldots, \mathcal{T}_{2k+1} \) are the initial conditions. Now we calculate the number of the distance-\( k \) independent dominating sets of \( P_n \). For \( k \geq 1 \), \( n \geq 2k+2 \) and \( 1 \leq i \leq k+1 \), let \( \mathcal{T}_n^{(i)} \) be the collection of all the distance-\( k \) independent dominating sets of \( \mathcal{T}_n \) which contain the vertex \( n-i+1 \). In the following lemma, we calculate the number \( i_k(P_n) \).
Lemma 2.3. For $n \geq 2k + 2$, we have the following results.
(i) For $1 \leq i_1 < i_2 \leq k + 1$, $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} = \emptyset$.
(ii) $i_k(P_n) = \sum_{i=1}^{k+1} i_k(P_{n-k-i})$.

Proof. (i) Suppose that $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} \neq \emptyset$, where $1 \leq i_1 < i_2 \leq k + 1$, and let $I \in \mathcal{T}_n^{(i_j)}$ for $j = 1$ and 2. Then we obtain that $n - i_1 + 1 \in I$ and $n - i_2 + 1 \in I$, by the definition, $i_2 - i_1 = (n - i_1 + 1) - (n - i_2 + 1) \geq k + 1$. This contradiction that $1 \leq i_1 < i_2 \leq k + 1$. Hence $\mathcal{T}_n^{(i_1)} \cap \mathcal{T}_n^{(i_2)} = \emptyset$ when $1 \leq i_1 < i_2 \leq k + 1$.

(ii) By (i) and Theorem 2.2, we have that $i_k(P_n) = |\mathcal{T}_n| = \sum_{i=1}^{k+1} |\mathcal{T}_n^{(i)}| = \sum_{i=1}^{k+1} |\mathcal{T}_{n-k-i}| = \sum_{i=1}^{k+1} i_k(P_{n-k-i})$. □

3 Instantiation

In this section, we characterize the sets $\mathcal{S}_3(P_n)$ for $1 \leq n \leq 15$. Besides, we calculate the numbers $i_k(P_n)$ for $1 \leq k \leq 10$ and $1 \leq n \leq 15$.

For $k = 3$, we show the sets $\mathcal{S}_3(P_1), \mathcal{S}_3(P_2), \ldots, \mathcal{S}_3(P_{15})$.

$\mathcal{S}_3(P_1) = \{\{1\}\}$.
$\mathcal{S}_3(P_2) = \{\{1\}, \{2\}\}$.
$\mathcal{S}_3(P_3) = \{\{1\}, \{2\}, \{3\}\}$.
$\mathcal{S}_3(P_4) = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.
$\mathcal{S}_3(P_5) = \{\{1, 5\}, \{2\}, \{3\}, \{4\}\}$.
$\mathcal{S}_3(P_6) = \{\{1, 6\}, \{2, 6\}, \{1, 5\}, \{3\}, \{4\}\}$.
$\mathcal{S}_3(P_7) = \{\{1, 7\}, \{2, 7\}, \{3, 7\}, \{1, 6\}, \{2, 6\}, \{1, 5\}, \{4\}\}$.
$\mathcal{S}_3(P_8) = \{\{1, 8\}, \{2, 8\}, \{3, 8\}, \{4, 8\}, \{1, 7\}, \{2, 7\}, \{3, 7\}, \{1, 6\}, \{2, 6\}, \{1, 5\}\}$.
$\mathcal{S}_3(P_9) = \{\{1, 5, 9\}, \{2, 9\}, \{3, 9\}, \{4, 9\}, \{1, 8\}, \{2, 8\}, \{3, 8\}, \{4, 8\},$
\{1, 7\}, \{2, 7\}, \{3, 7\}, \{1, 6\}, \{2, 6\}\}.
$\mathcal{S}_3(P_{10}) = \{\{1, 6, 10\}, \{2, 6, 10\}, \{1, 5, 10\}, \{3, 10\}, \{4, 10\}, \{1, 5, 9\}, \{2, 9\}, \{3, 9\},$
\{4, 9\}, \{1, 8\}, \{2, 8\}, \{3, 8\}, \{4, 8\}, \{1, 7\}, \{2, 7\}, \{3, 7\}\}.
$\mathcal{S}_3(P_{11}) = \{\{1, 7, 11\}, \{2, 7, 11\}, \{3, 7, 11\}, \{1, 6, 11\}, \{2, 6, 11\}, \{1, 5, 11\}, \{4, 11\}\}.$
$\mathcal{S}_3(P_{12}) = \{\{1, 7, 12\}, \{2, 7, 12\}, \{1, 6, 12\}, \{3, 8, 12\}, \{4, 8, 12\}, \{1, 7, 12\}, \{2, 7, 12\},$
\{3, 7, 12\}, \{1, 6, 12\}, \{2, 6, 12\}, \{1, 5, 12\}, \{1, 7, 11\}, \{2, 7, 11\},$
\{3, 7, 11\}, \{1, 6, 11\}, \{2, 6, 11\}, \{1, 5, 11\}, \{4, 11\}\}.$
\[ \mathcal{I}_{3}(P_{13}) = \{\{1, 5, 9, 13\}, \{2, 9, 13\}, \{3, 9, 13\}, \{4, 9, 13\}, \{1, 8, 13\}, \{2, 8, 13\}, \{3, 8, 13\}, \{4, 8, 13\}, \{1, 7, 13\}, \{2, 7, 13\}, \{3, 7, 13\}, \{1, 6, 13\}, \{2, 6, 13\}, \{1, 8, 12\}, \{2, 8, 12\}, \{3, 8, 12\}, \{4, 8, 12\}, \{1, 7, 12\}, \{2, 7, 12\}, \{3, 7, 12\}, \{1, 6, 12\}, \{2, 6, 12\}, \{1, 5, 12\}, \{1, 7, 11\}, \{2, 7, 11\}, \{3, 7, 11\}, \{1, 6, 11\}, \{2, 6, 11\}, \{1, 5, 11\}, \{4, 11\}\}, \{1, 6, 10\}, \{2, 6, 10\}, \{1, 5, 10\}, \{3, 10\}, \{4, 10\}\}.
\]

\[ \mathcal{I}_{3}(P_{14}) = \{\{1, 6, 10, 14\}, \{2, 6, 10, 14\}, \{1, 5, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}, \{1, 5, 9, 14\}, \{2, 9, 14\}, \{3, 9, 14\}, \{4, 9, 14\}, \{1, 8, 14\}, \{2, 8, 14\}, \{3, 8, 14\}, \{4, 8, 14\}, \{1, 7, 14\}, \{2, 7, 14\}, \{3, 7, 14\}, \{1, 5, 13\}, \{2, 5, 13\}, \{2, 9, 13\}, \{3, 9, 13\}, \{4, 9, 13\}, \{1, 8, 13\}, \{2, 8, 13\}, \{3, 8, 13\}, \{4, 8, 13\}, \{1, 7, 13\}, \{2, 7, 13\}, \{3, 7, 13\}, \{1, 5, 12\}, \{1, 7, 11\}, \{2, 7, 11\}, \{3, 7, 11\}, \{1, 6, 11\}, \{2, 6, 11\}, \{1, 5, 11\}, \{4, 11\}\}.
\]

\[ \mathcal{I}_{3}(P_{15}) = \{\{1, 7, 11, 15\}, \{2, 7, 11, 15\}, \{3, 7, 11, 15\}, \{1, 6, 11, 15\}, \{2, 6, 11, 15\}, \{1, 5, 11, 15\}, \{4, 11, 15\}, \{1, 6, 10, 15\}, \{2, 6, 10, 15\}, \{1, 5, 10, 15\}, \{3, 10, 15\}, \{4, 10, 15\}, \{1, 5, 9, 15\}, \{2, 9, 15\}, \{3, 9, 15\}, \{4, 9, 15\}, \{1, 8, 15\}, \{2, 8, 15\}, \{3, 8, 15\}, \{4, 8, 15\}, \{1, 6, 10, 14\}, \{2, 6, 10, 14\}, \{1, 5, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}, \{1, 5, 9, 14\}, \{2, 9, 14\}, \{3, 9, 14\}, \{4, 9, 14\}, \{1, 8, 14\}, \{1, 8, 14\}, \{2, 8, 14\}, \{3, 8, 14\}, \{4, 8, 14\}, \{1, 7, 14\}, \{2, 7, 14\}, \{3, 7, 14\}, \{1, 5, 13\}, \{2, 5, 13\}, \{2, 9, 13\}, \{3, 9, 13\}, \{4, 9, 13\}, \{1, 8, 13\}, \{2, 8, 13\}, \{3, 8, 13\}, \{4, 8, 13\}, \{1, 7, 13\}, \{2, 7, 13\}, \{3, 7, 13\}, \{1, 6, 13\}, \{2, 6, 13\}, \{1, 5, 12\}, \{2, 7, 12\}, \{3, 7, 12\}, \{1, 6, 12\}, \{2, 6, 12\}, \{1, 5, 12\}\}.
\]

We end with the following table for summarizing the numbers \(i_{k}(P_{n})\) for

\[ 1 \leq k \leq 10 \text{ and } 1 \leq n \leq 15. \]
Table 1: The numbers $i_k(P_n)$ for $1 \leq k \leq 10$ and $1 \leq n \leq 15.$

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