Top–Designs in the Category of Fort Spaces

Mehrnaz Pourattar

Islamic Azad University, Science and Research Branch, Tehran, Iran

Fatemah Ayatollah Zadeh Shirazi

Faculty of Mathematics, Statistics and Computer Science
College of Science, University of Tehran
Enghelab Ave., Tehran, Iran

Abstract

In infinite topological Fort space $X$, for nonempty subsets $C, D$ of $X$ in the following text we answer to this question “Is there any $\lambda$ and Top–design $C - (X, D, \lambda)$ of type $i$?” for $i = 1, 2, 3, 4$. We prove there exist $\lambda$ and $C - (X, D, \lambda)$, Top–design of type 2 (resp. type 4) if and only if $C$ can be embedded into $D$.

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1 Introduction

Suppose $S$ is a finite set with $n \geq 2$ elements (so $S$ is an $n$–set) and $A$ is a collection of $k$–subsets of $S$ such that each $t$–subset of $S$ occurs exactly in $\lambda$ elements of $A$, then $A$ is favorit and well studied traditional $t - (n, k, \lambda)$ combinatorial design ($t < n$ and $\lambda \geq 1$) (see [1, 3]). However these finite traditional designs has been generalized in “infinite designs” in [3], also generalized designs have been introduced for the first time in [2] as a generalization
of combinatorial designs in different mathematical categories like category of well-ordered sets, topological spaces, etc.. We use term Top-design when our reference category is the category of topological spaces.

Using the same notations as in [2], in topological space $X$ for nonempty subsets $C, D$ of $X$, nonzero cardinal number $\lambda$ and collection $A$ of subsets of $X$ using statements (where by $S \approx T$ we mean $S$ and $T$ are homeomorphic spaces):

I. $\forall B \in A \ (B \approx D)$

II. $\forall B \in A \ (B \approx D \land X \setminus B \approx X \setminus D)$

III. $\forall E \subseteq X \ (E \approx C \Rightarrow \text{card}({B \in A : E \subseteq B}) = \lambda)$

IV. $\forall E \subseteq X \ ((E \approx C \land X \setminus E \approx X \setminus C) \Rightarrow \text{card}({B \in A : E \subseteq B}) = \lambda)$,

we say $A$ is a :

- $C - (X, D, \lambda)$ Top-design of type 1, if (II) and (III)
- $C - (X, D, \lambda)$ Top-design of type 2, if (I) and (III)
- $C - (X, D, \lambda)$ Top-design of type 3, if (II) and (IV)
- $C - (X, D, \lambda)$ Top-design of type 4, if (I) and (IV)

Let’s mention that if $b \in X$, equip $X$ with topology $\{U \subseteq X : b \notin U \lor (X \setminus U \text{ is finite})\}$, then we say $X$ is a Fort space with particular point $b$ [4, Counterexample 24]. One may find counterexamples regarding $C - (\{\frac{1}{n} : n \geq 1\} \cup \{0\}, D, \lambda)$ Top-designs in [2], note to the fact that $\{\frac{1}{n} : n \geq 1\} \cup \{0\}$ (with induced topology of $\mathbb{R}$) is an infinite countable Fort space, leads us to study other types of infinite Fort spaces in the approach of Top-designs.

**Note 1.1.** Two Fort spaces are homeomorphic if and only if they are in one-to-one correspondence. Moreover in Fort space $X$ with particular point $b$ infinite subset $Y$ of $X$ as subspace topology has Fort topology if and only if $b \in Y$ (all finite subsets of $X$ are finite discrete spaces and carry Fort topology structure).

**Convention 1.2.** In the following text suppose $X$ is an infinite Fort space with the particular point $b$.

## 2 Results in Top-designs on $X$

In this section we study the existence of $C - (X, D, \lambda)$ for different $Cs$ and $Ds$.

**Lemma 2.1.** For $U, V \subseteq X$ with $U \approx V$ and $X \setminus U \approx X \setminus V$ we have:
1. $b \in U$ if and only if $b \in V$ (i.e., $U \cap \{b\} = V \cap \{b\}$),

2. for infinite $U$ with $\text{card}(U) < \text{card}(X)$ and $H \subseteq X$ we have $U \approx H$ and $X \setminus U \approx X \setminus H$ if and only if $\text{card}(U) = \text{card}(H)$ and $U \cap \{b\} = H \cap \{b\}$.

Proof. 1) First suppose $U$ is infinite, so $V$ is infinite too. Since $b$ is the unique limit point of any infinite subset of $X$, $U$ contains a limit point if and only if $b \in U$ on the other hand $U$ contains a limit point if and only if $V$ contains a limit point which means $b \in V$ in its turn.

Now suppose $U$ is finite, thus $X \setminus U$ is infinite and using a similar method described above, we have $b \in X \setminus U$ if and only if $b \in X \setminus V$ which completes the proof.

2) Suppose $\text{card}(U) = \text{card}(H)$ and $U \cap \{b\} = H \cap \{b\}$, then $\text{card}(U \setminus \{b\}) = \text{card}(H \setminus \{b\})$, thus there exists bijection $f : U \setminus \{b\} \rightarrow H \setminus \{b\}$. If $b \notin U$, then $b \notin H$ and $f : U \setminus \{b\} = U \rightarrow H \setminus \{b\} = H$ is a homeomorphism (of discrete spaces) too. If $b \in U$, then $b \in H$ too and $\tilde{f} : U \rightarrow H$ with $\tilde{f} |_{U \setminus \{b\}} = f$ and $\tilde{f}(b) = b$ is a homeomorphism of infinite Fort spaces ($U$ and $H$ with particular point $b$). So $U \approx H$.

On the other hand if $\text{card}(U) = \text{card}(H) < \text{card}(X)$, then $\text{card}(X \setminus U) = \text{card}(X \setminus H) = \text{card}(X)$. Also if $U \cap \{b\} = H \cap \{b\}$, then $(X \setminus U) \cap \{b\} = (X \setminus H) \cap \{b\}$. So if $\text{card}(U) = \text{card}(H) < \text{card}(X)$ and $U \cap \{b\} = H \cap \{b\}$, then $\text{card}(X \setminus U) = \text{card}(X \setminus H)$ and $(X \setminus U) \cap \{b\} = (X \setminus H) \cap \{b\}$ which shows $X \setminus U \approx X \setminus H$ by the above argument.

Use item (1) to complete the proof of (2). \hfill \Box

Note that if there exists a $C = (X, D, \lambda)$, Top–design of type $i$, then there exists $U \approx D$ with $C \subseteq U$, so $\text{card}(C) \leq \text{card}(U) = \text{card}(D)$. Therefore $\text{card}(C) = \min(\text{card}(D), \text{card}(C)) \leq \text{card}(X)$.

Theorem 2.2. Regarding 1st type of Top–designs for nonempty subsets $C, D$ of $X$ we have:

a. suppose $b \notin C \cup D$:

a1. if $C$ is finite, then there is not any $C = (X, D, \lambda)$, Top–design of type 1,

a2. if $C$ is infinite and $\text{card}(C) < \text{card}(X)$, then there exist $\lambda$ and a $C = (X, D, \lambda)$ Top–design of type 1,

a3. if $\text{card}(C) = \text{card}(D) = \text{card}(X)$, then there exist $\lambda$ and a $C = (X, D, \lambda)$ Top–design of type 1 if and only if $D = X \setminus \{b\}$,

b. if $b \in C \setminus D$, then there is not any $C = (X, D, \lambda)$, Top–design of type 1,

c. suppose $b \in D$:

$\text{c1. for finite } C \text{ there exist } \lambda \text{ and a } C = (X, D, \lambda) \text{ Top–design of type 1 if and only if } \text{card}(C) + 2 \leq \text{card}(D)$,
c₂. if $C$ is infinite and $\text{card}(C) = \min(\text{card}(D), \text{card}(C)) < \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top–design of type 1,

c₃. if $\text{card}(C) = \text{card}(D) = \text{card}(X)$, then there exist $\lambda$ and a $C - (X, D, \lambda)$ Top–design of type 1 if and only if $D = X$.

Proof. Let $\mathcal{W} = \{E \subseteq X : E \approx D \cap X \setminus E \approx X \setminus D\}$. By item (1) in Lemma 2.1, it’s evident that $b \in D$ if and only if $b \in \bigcup \mathcal{W}$ (resp. $b \in \bigcap \mathcal{W}$).

a₁) Choose $k \in C$, if $A$ is a $C - (X, D, \lambda)$, Top–design of type 1, then $A \subseteq \mathcal{W}$ and $A$ is a $(C \setminus \{k\}) \cup \{b\} - (X, D, \lambda)$, Top–design of type 1 too, which is a contradiction since $b \not\in \bigcup \mathcal{W}$.

a₂) We have the following sub–cases:

- $\text{card}(C) \leq \text{card}(D) < \text{card}(X)$. In this case by item (2) in Lemma 2.1 we have $\mathcal{W} = \{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(D)\}$. Using $\text{card}(X \setminus \{b\}) = \text{card}(X \setminus (C \cup \{b\}))$ for $\mathcal{W}' = \{E \subseteq X \setminus (C \cup \{b\}) : \text{card}(E) = \text{card}(D)\}$ we have $\text{card}(\mathcal{W}) = \text{card}(\mathcal{W}')$. It’s evident that $\mathcal{W}' \to \mathcal{W}$ is one–to–one, so $\text{card}(\mathcal{W}') \leq \text{card}(\{F \in \mathcal{W} : C \subseteq F\}) \leq \text{card}(\mathcal{W})$. Thus $\text{card}(\{F \in \mathcal{W} : C \subseteq F\}) = \text{card}(\mathcal{W})$. Since $C$ is infinite and $b \not\in C$, for all subset $E$ of $X$ with $C \approx E$ we have $\text{card}(C) = \text{card}(E)$ and $b \not\in E$, so by a similar method described for $C$ we have $\text{card}(\{F \in \mathcal{W} : E \subseteq F\}) = \text{card}(\mathcal{W})$. Hence $\mathcal{W}$ is a $C - (X, D, \text{card}(\mathcal{W}))$ Top–design of type 1.

- $\text{card}(C) < \text{card}(D) = \text{card}(X)$. In this case by Lemma 2.1, $\mathcal{W} = \{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(D) \land \text{card}(X \setminus E) = \text{card}(X \setminus D)\}$. Since $\text{card}(C) < \text{card}(D)$ and $C, D$ carry discrete topologies thus $C$ can be embedded in $D$ and without any loss of generality we may suppose $C \subseteq D$. By infiniteness of $D$, at least one of the sets $D \setminus C$ or $C$ is infinite and

$$\text{card}(C) < \text{card}(X) = \text{card}(D) = \text{card}(C) + \text{card}(D \setminus C) = \max(\text{card}(C), \text{card}(D \setminus C))$$

so we have $\max(\text{card}(C), \text{card}(D \setminus C)) = \text{card}(D \setminus C) = \text{card}(X)$. Since $2\text{card}(D \setminus C) = \text{card}(D \setminus C)$, we may choose $H \subseteq D \setminus C$ with

$$\text{card}(H) = \text{card}(D \setminus C) \setminus H = \text{card}(D \setminus (C \cup H)) = \text{card}(D \setminus C) = \text{card}(X)$$

Let $K = \{F \subseteq D \setminus (H \cup C) : \text{card}(F \cup \{b\}) = \text{card}(X \setminus D)\}$, and consider the following claim:

Claim. For $F \in K$ we have $C \subseteq X \setminus (F \cup \{b\}) \in \mathcal{W}$. Suppose $F \in K$, so $F \subseteq D \setminus (H \cup C)$ so $H \subseteq X \setminus (F \cup \{b\}) \subseteq X$ thus $\text{card}(X \setminus (F \cup \{b\})) = \text{card}(X) = \text{card}(D)$ and $\text{card}(X \setminus D) = \text{card}(F \cup \{b\}) = \text{card}(X \setminus (X \setminus (F \cup \{b\})))$, therefore $X \setminus (F \cup \{b\}) \in \mathcal{W}$. Also $F \subseteq D \setminus (H \cup C)$ and $b \not\in C$ show $C \subseteq X \setminus (F \cup \{b\})$. Therefore

$$\eta : K \to \{B \in \mathcal{W} : C \subseteq B\}$$
is well-defined and clearly one-to-one.
Thus $\text{card}(K) \leq \text{card}((\{B \in \mathcal{W} : C \subseteq B\}) \leq \text{card}(\mathcal{W})$, however using $\text{card}(D \setminus (H \cup C)) = \text{card}(X \setminus \{b\})$ we have:

$$
\text{card}(\mathcal{W}) \leq \text{card}(\{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(X \setminus (D \cup \{b\}))\}) \\
= \text{card}(\{F \subseteq D \setminus (H \cup C) : \text{card}(F) = \text{card}(X \setminus (D \cup \{b\}))\}) \\
= \text{card}(K)
$$

which leads to $\text{card}(K) = \text{card}(\{B \in \mathcal{W} : C \subseteq B\}) = \text{card}(\mathcal{W})$.

For $E \subseteq X$ with $E \approx C$ (so $b \notin E$), we have $D' = (D \setminus C) \cup E \in \mathcal{W}$ and $\mathcal{W} = \{E \subseteq X \setminus \{b\} : \text{card}(E) = \text{card}(D') \land \text{card}(X \setminus E) = \text{card}(X \setminus D')\}$. Using a similar method described above, we have $\text{card}(\{B \in \mathcal{W} : E \subseteq B\}) = \text{card}(\mathcal{W})$, thus $\mathcal{W}$ is a $C - (X, D, \text{card}(\mathcal{W}))$ Top–design of type 1.

$a_2)$ In this case if $\mathcal{A}$ is a $C - (X, D, \lambda)$ Top–design of type 1, then there exists $B \in \mathcal{A}$ with $X \setminus \{b\} \subseteq B$ (since $C \approx X \setminus \{b\}$), which leads to $D = X \setminus \{b\}$, and $\mathcal{W} = \{X \setminus \{b\}\} \setminus (X, D, 1)$ Top–design of type 1.

$b)$ Use the fact that if $b \notin D$, then for all $B \subseteq X$ with $D \approx B$ and $X \setminus D \approx X \setminus B$ we have $b \notin B$, and in particular $C \notin B$.

$c_1)$ First suppose $\mathcal{A}$ is a $C - (X, D, \lambda)$ Top–design of type 1 and $D$ is finite, then for all subsets $H$ of $X$ with $\text{card}(H) = \text{card}(C)$, there exists $B \in \mathcal{A}$ with $H \subseteq B$, however we may assume $b \notin H$, using $b \in B$ we have $\text{card}(H) \leq \text{card}(B \setminus \{b\}) = \text{card}(D) - 1$. Hence $\text{card}(C) + 1 \leq \text{card}(D)$. If $\text{card}(C) + 1 = \text{card}(D)$ then any subset of $X \setminus \{b\}$ with $\text{card}(C)$ elements occurs in just one element of $\mathcal{A}$ and $\mathcal{A} = \{S \setminus \{b\} : S \subseteq X \setminus \{b\} \land \text{card}(S) = \text{card}(C)\}$ now choose a subset $J$ of $X \setminus \{b\}$ with $\text{card}(C) - 1$ elements, then infinite elements of $\mathcal{A}$ contain $J \cup \{b\}$ which is in contradiction with $\lambda = 1$, so $\text{card}(C) + 1 < \text{card}(D)$ and $\text{card}(C) + 2 \leq \text{card}(D)$.

In order to complete the proof, we have the following cases:

Case 1. $X$ is uncountable and $D$ is infinite. In this case choose infinite countable subset $I$ of $D \setminus \{b\}$. By the proof of (a$_2$) for

$$
\mathcal{W}_{-b} = \{E \subseteq X : E \approx D \setminus \{b\} \land X \setminus E \approx X \setminus (D \setminus \{b\})\}
$$

is a $I - (X, D \setminus \{b\}, \text{card}(\mathcal{W}_{-b}))$ Top–design of first type. We show $\mathcal{W}$ is a $C - (X, D, \text{card}(\mathcal{W}))$ Top–design of first type. Consider $H \subseteq X$ with $H \approx C$.

There exists $J \subseteq X \setminus \{b\}$ with $H \setminus \{b\} \subseteq J$ and $J \approx I$ so

$$
\text{card}(\mathcal{W}_{-b}) \geq \text{card}(\{B \in \mathcal{W}_{-b} : H \setminus \{b\} \subseteq B\}) \\
\geq \text{card}(\{B \in \mathcal{W}_{-b} : J \subseteq B\}) = \text{card}(\mathcal{W}_{-b})
$$

therefore $\text{card}(\{B \in \mathcal{W}_{-b} : H \setminus \{b\} \subseteq B\}) = \text{card}(\mathcal{W}_{-b})$. Considering bijection $\eta : \mathcal{W}_{-b} \to \mathcal{W}$, and $b \in \bigcap \mathcal{W}$ we have $\text{card}(\{B \in \mathcal{W} : H \setminus \{b\} \subseteq B\}) = \text{card}(\mathcal{W}_{-b} : b \in B \cup \{b\}) = \text{card}(\{B \in \mathcal{W}_{-b} : H \setminus \{b\} \subseteq B\}) = \text{card}(\mathcal{W}_{-b}) = \text{card}(\mathcal{W})$ which leads to
\[ \text{card}\{B \in \mathcal{W} : H \subseteq B\} = \text{card}(\mathcal{W}) \text{ and } \mathcal{W} \text{ is a } C - (X, D, \text{card}(\mathcal{W})) \text{ Top–design of first type.} \]

**Case 2.** \( X, D \) and \( X \setminus D \) are infinite countable. In this case we may suppose \( X \setminus \{b\} = \{p_n : n \geq 1\} \) and \( D = \{p_{2n} : n \geq 1\} \cup \{b\} \) with distinct \( p_n \)s. Let \( \mathcal{A} = \{X \setminus \{p_{2k+1} : k \geq s\} : s \geq 1\} \), then \( \mathcal{A} \) is a \( C - (X, D, \aleph_0) \) Top–design of type 1.

**Case 3.** \( X \) and \( D \) are infinite countable and \( X \setminus D \neq \emptyset \) is finite. In this case \( \mathcal{W} \) is infinite countable and a \( C - (X, D, \aleph_0) \) Top–design of type 1.

**Case 4.** \( X = D \) is infinite countable. In this case \( \mathcal{W} = \{X\} \) is a \( C - (X, D, 1) \) Top–design of type 1.

**Case 5.** \( D \) is finite and \( \text{card}(C) + 2 \leq \text{card}(D) \). In this case \( \text{card}(\mathcal{W}) = \text{card}(X) \) (since for infinite set \( X \) we have \( \text{card}(X) = \text{card}(\mathcal{P}_{\text{fin}}(X)) \), where \( \mathcal{P}_{\text{fin}}(X) \) is the collection of all finite subsets of \( X \) and \( \mathcal{W} \) is a \( C - (X, D, \text{card}(X)) \) Top–design of type 1.

\( c_2 \) In this case by the proof of \((a_2)\), \( \mathcal{W} \) is a \( C \setminus \{b\} - (X, D \setminus \{b\}, \text{card}(\mathcal{W})) \) Top–design of type 1, using \( b \in \bigcap \mathcal{W} \), shows that \( \mathcal{W} \) is a \( C - (X, D, \text{card}(\mathcal{W})) \) Top–design of type 1 too.

\( c_2 \) Use a similar method described in the proof of \((a_3)\).

**Lemma 2.3.** For nonempty subsets \( C, D \) of \( X \), \( C \) can be embedded into \( D \) if and only if

"\( C \) is finite or \( b \notin C \setminus D \), and \( \text{card}(C) \leq \text{card}(D) \)."

**Proof.** Suppose \( C \) can be embedded in \( D \) and choose \( E \subseteq D \) with \( E \cong C \), so \( \text{card}(C) = \text{card}(E) \leq \text{card}(D) \). If \( C \) is infinite and \( b \in C \) then any subset of \( X \) homeomorphic with \( C \) contains \( b \), thus \( b \in E(\subseteq D) \) and \( b \notin C \setminus D \). \( \square \)

**Theorem 2.4.** For nonempty subsets \( C, D \) of \( X \), there exist \( \lambda \) and a \( C - (X, D, \lambda) \), Top–design of type 2 if and only if \( C \) can be embedded into \( D \).

**Proof.** If we can not embed \( C \) into \( D \) it’s evident that there is not any \( C - (X, D, \lambda) \), Top–design of type 2.

Conversely suppose \( C \) can be embedded in \( D \), so by Lemma 2.3 \( \text{card}(C) \leq \text{card}(D) \) and "\( C \) is finite or \( b \notin C \setminus D \)." Let \( \mathbb{L} = \{E \subseteq X : E \cong D\} \). We have the following cases:

- \( \text{card}(C) \leq \text{card}(D) \) and \( C \) is finite. In this case \( \mathbb{L} \) is a \( C - (X, D, \lambda) \) Top–design of type 2 with:

\[
\lambda = \begin{cases} 
1 & \text{card}(C) = \text{card}(D), \\
\text{card}(\{E \subseteq X : \text{card}(E) = \text{card}(D)\}) & \text{otherwise}.
\end{cases}
\]

For this aim use the fact that \( \eta : \{E \subseteq X \setminus C : \text{card}(E) = \text{card}(D)\} \rightarrow \{E \subseteq X : \text{card}(E) = \text{card}(D)\} \) with \( \eta(E) = E \cup C \) is bijective.

- \( \text{card}(C) = \min(\text{card}(C), \text{card}(D)) < \text{card}(X) \) and \( b \notin C \setminus D \). In this case by
Theorem 2.2 there exists $\lambda$ and $C - (X, D, \lambda)$, Top–design of type 1, so it is a $C - (X, D, \lambda)$, Top–design of type 2 too.

- $\text{card}(C) = \min(\text{card}(C), \text{card}(D)) = \text{card}(X)$ and $b \notin C \setminus D$. In this case $A = \{(X \setminus \{b\}) \cup (D \cap \{b\})\}$ is a $C - (X, D, 1)$ Top–design of type 2.

**Theorem 2.5.** Regarding 3rd type of Top–designs for nonempty subsets $C, D$ of $X$, there exist $\lambda$ and a $C - (X, D, \lambda)$, Top–design of type 3 if and only if $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$.

**Proof.** Let $W = \{E \subseteq X : E \approx D \cap X \setminus E \approx X \setminus D\}$. If $A$ is a $C - (X, D, \lambda)$, Top–design of type 3, then $A \subseteq W$ and we have the following cases:

**Case 1.** $b \in C \setminus D$. In this case for all $E \in A(\subseteq W)$, we have $b \notin E$ and $C \not\subseteq E$ thus $A$ is not a $C - (X, D, \lambda)$, Top–design of type 3.

**Case 2.** $\text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\})$, and “$b \in C \cap D$ or $b \notin C \cup D$”. In this case we have $\text{card}(C) > \text{card}(D)$ so we can not embed $C$ into $D$ and it’s evident that there is not any $C - (X, D, \lambda)$, Top–design of type 3.

**Case 3.** $\text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\})$, $b \in D \setminus C$. In this case for all $B \in A$, $b \in B$ and $\text{card}(C) = \text{card}(C \setminus \{b\}) > \text{card}(D \setminus \{b\}) = \text{card}(B \setminus \{b\})$ so $C \not\subseteq B \setminus \{b\}$ and $C \not\subseteq B$ so $A$ is not a $C - (X, D, \lambda)$, Top–design of type 3.

**Case 4.** $\text{card}(X \setminus (D \cup \{b\})) > \text{card}(X \setminus (C \cup \{b\}))$. In this case for all $E \in W$ we have $\text{card}(X \setminus (E \cup \{b\})) > \text{card}(X \setminus (C \cup \{b\}))$, thus $X \setminus (E \cup \{b\}) \not\subseteq X \setminus (C \cup \{b\})$ and $C \not\subseteq E$, so there is not any $C - (X, D, \lambda)$, Top–design of type 3.

Considering the above cases $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$.

Conversely, suppose $b \notin C \setminus D$, $\text{card}(C \setminus \{b\}) \leq \text{card}(D \setminus \{b\})$ and $\text{card}(X \setminus (D \cup \{b\})) \leq \text{card}(X \setminus (C \cup \{b\}))$, then $W$ is a $C - (X, D, \lambda)$, Top–design of type 3 for $\lambda = \text{card}(\{E \in W : C \subseteq E\})$ (note that for $F \subseteq X$ with $F \approx C$ and $X \setminus F \approx X \setminus C$, the map $\{E \in W : F \subseteq E\} \rightarrow \{E \in W : C \subseteq E\}$ is bijective).

**Theorem 2.6.** For nonempty subsets $C, D$ of $X$, there exist $\lambda$ and a $C - (X, D, \lambda)$, Top–design of type 4 if and only if $C$ can be embedded into $D$.

**Proof.** If $C$ can be embedded into $D$, then there exist $\lambda > 0$ and a $C - (X, D, \lambda)$ Top–design of type 2 like $A$ by Theorem 2.4, so $A$ is a $C - (X, D, \lambda)$ Top–design of type 4 too.

Conversely, it’s evident that if $A$ is a Top–design of type $i$ (for $i = 1, 2, 3, 4$), then there exists $E \in A$ with $C \subseteq E$, using $E \approx D$ leads us to the fact that $C$ can be embedded into $D$.

**Theorem 2.7.** For nonempty subsets $C, D$ of $X$ the following statements are equivalent:
• there is not any $C - (X, D, \lambda)$, Top–design of type 2,
• there is not any $C - (X, D, \lambda)$, Top–design of type 4,
• “$C$ is infinite and $b \in C \setminus D$”, or “$\text{card}(C) > \text{card}(D)$”,
• $C$ can not be embedded into $D$.

Proof. Theorems 2.4, 2.6 and Lemma 2.3. \qed

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