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# Symmetric Properties for the Second Kind Generalized $(h, q)$-Euler Polynomials

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**Abstract**

In this paper, we study the symmetry for the second kind generalized $(h, q)$-Euler numbers $E_{n,\chi,q}^{(h)}$ and polynomials $E_{n,\chi,q}^{(h)}(x)$. We obtain some interesting identities of the power odd sums and the second kind generalized $(h, q)$-Euler polynomials $E_{n,\chi,q}^{(h)}(x)$ using the symmetric properties for the $p$-adic invariant integral on $\mathbb{Z}_p$.

**Mathematics Subject Classification:** 11B68, 11S40, 11S80

**Keywords:** Euler numbers and polynomials of the second kind, the generalized Euler numbers and polynomials of the second kind, symmetric properties, power odd sums

## 1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If

$q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [2] }.$$

Note that

$$\lim_{q\to 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \tag{1.2}$$

Let a fixed positive integer $d$ with $(p,d)=1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$.

It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.3}$$

We assume that $h \in \mathbb{Z}$. First, we introduced the second kind Euler numbers and Euler polynomials(cf. [3, 4, 5, 6]). We investigated the zeros of the second kind Euler polynomials $E_n(x)$. The second kind Euler numbers $E_n$ are defined by the generating function:

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}),$$

where we use the technique method notation by replacing $E^n$ by $E_n (n \ge 0)$ symbolically. We consider the second kind Euler polynomials $E_n(x)$ as follows:

$$\left(\frac{2e^t}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

Note that $E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k x^{n-k}$. In the special case $x = 0$, we define $E_n(0) = E_n$.

Now, we construct the second kind generalized $(h,q)$-Euler numbers $E_{n,\chi,q}^{(h)}$ and polynomials $E_{n,\chi,q}^{(h)}(x)$ attached to $\chi$. Let $\chi$ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. The second kind generalized $(h,q)$-Euler numbers $E_{n,\chi,q}^{(h)}$ attached to $\chi$ are defined by the generating function:

$$\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a q^{ha} e^{(2a+1)t}}{q^{hd}e^{2dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)} \frac{t^n}{n!}. \tag{1.5}$$

We consider the second kind generalized $(h,q)$-Euler polynomials $E_{n,\chi,q}^{(h)}(x)$ attached to $\chi$ as follows:

$$\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a q^{ha} e^{(2a+1)t}}{q^{hd}e^{2dt}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.6}$$

When $\chi = \chi^0$ and $q \to 1$, above (1.5) and (1.6) will become the corresponding definitions of the second kind Euler numbers and Euler polynomials, respectively.

Let $g(y) = \chi(y) q^{hy} e^{(2y+1+x)t}$. By (1.3), we derive

$$\begin{aligned} I_{-1}\left(\chi(y) q^{hy} e^{(2y+1+x)t}\right) &= \int_X \chi(y) q^{hy} e^{(2y+1+x)t} d\mu_{-1}(y) \\ &= \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a q^{ha} e^{(2a+1)t}}{q^{hd}e^{2dt}+1} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

By using Taylor series of $e^{(2y+1+x)t}$ in the above equation (1.7), we obtain

$$\sum_{n=0}^{\infty} \left( \int_X \chi(y) q^{hy} (2y+1+x)^n d\mu_{-1}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\dfrac{t^n}{n!}$ in the above equation, we have the Witt formula for the second kind generalized $(h,q)$- Euler polynomials attached to $\chi$ as follows:

**Theorem 1.1** *For positive integers $n$ and $h \in \mathbb{Z}$, we have*

$$E_{n,\chi,q}^{(h)}(x) = \int_X \chi(y) q^{hy} (2y+1+x)^n d\mu_{-1}(y). \tag{1.8}$$

If we take $x = 0$ in Theorem 1.1, we also obtain the following corollary.

**Corollary 1.2** *For positive integers $n$ and $h \in \mathbb{Z}$, we have*

$$E^{(h)}_{n,\chi,q} = \int_X \chi(y) q^{hy} (2y+1)^n d\mu_{-1}(y). \tag{1.9}$$

From (1.8) and (1.9), we have the following theorem.

**Theorem 1.3** *For positive integers $n$ and $h \in \mathbb{Z}$, we have*

$$E^{(h)}_{n,\chi,q}(x) = \sum_{l=0}^{n} \binom{n}{l} E^{(h)}_{l,\chi,q}.$$

# 2 Symmetry for for the second kind generalized $(h, q)$-Euler polynomials

In this section, we obtain some interesting identities of the power odd sums and the second kind generalized polynomials $E^{(h)}_{n,\chi,q}(x)$ using the symmetric properties for the $p$-adic invariant integral on $\mathbb{Z}_p$. We assume that $q \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. If $n$ is odd from (1.2), we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^k g(k). \tag{2.1}$$

It will be more convenient to write (2.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2 \sum_{k=0}^{n-1} (-1)^k g(k). \tag{2.2}$$

Substituting $g(x) = \chi(x) q^{hx} e^{(2x+1)t}$ into the above, we have

$$\begin{aligned} &\int_X \chi(x+n) q^{h(x+n)} e^{(2(x+n)+1)t} d\mu_{-1}(x) + \int_X \chi(x) q^{hx} e^{(2x+1)t} d\mu_{-1}(x) \\ &= 2 \sum_{j=0}^{n-1} (-1)^j \chi(j) q^{hj} e^{(2j+1)t}. \end{aligned} \tag{2.3}$$

For $k \in \mathbb{Z}_+$, let us define the power odd sums $T^{(h)}_{k,\chi,q}(n)$ as follows:

$$T^{(h)}_{k,\chi,q}(n) = \sum_{l=0}^{n} (-1)^l \chi(l) q^{hl} (2l+1)^k. \tag{2.4}$$

After some elementary calculations, we have

$$\int_X \chi(x)q^{hx}e^{(2x+1)t}d\mu_{-1}(x) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^{ha}e^{(2a+1)t}}{q^{hd}e^{2dt}+1},$$
$$\int_X \chi(x)q^{h(x+n)}e^{(2(x+n)+1)t}d\mu_{-1}(x) = q^{hn}e^{2nt}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^{ha}e^{(2a+1)t}}{q^{hd}e^{2dt}+1}. \tag{2.5}$$

By using (2.5), we obtain

$$\int_X \chi(x)q^{h(x+nd)}e^{(2(x+nd)+1)t}d\mu_{-1}(x) + \int_X \chi(x)q^{hx}e^{(2x+1)t}d\mu_{-1}(x)$$
$$= \left(1+q^{hnd}e^{2ndt}\right)\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^{ha}e^{(2a+1)t}}{q^{hd}e^{2dt}+1}.$$

From the above, we have

$$\int_X \chi(x)q^{h(x+nd)}e^{(2(x+nd)+1)t}d\mu_{-1}(x) + \int_X \chi(x)q^{hx}e^{(2x+1)t}d\mu_{-1}(x)$$
$$= \frac{2\int_X \chi(x)q^{hx}e^{(2x+1)t}d\mu_{-1}(x)}{\int_X q^{hndx}e^{2ndtx}d\mu_{-1}(x)}. \tag{2.6}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (2.3), we obtain

$$\sum_{m=0}^{\infty}\left(\int_X \chi(x)q^{h(x+nd)}(2x+1+2nd)^m d\mu_{-1}(x) + \int_X \chi(x)q^{hx}(2x+1)^m d\mu_{-1}(x)\right)\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{nd-1}(-1)^j\chi(j)q^{hj}(2j+1)^m\right)\frac{t^m}{m!}.$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)q^{hx}(2x+1)^k d\mu_{-1}(x)$$
$$+\int_X \chi(x)q^{hx}(2x+1)^m d\mu_{-1}(x) = 2\sum_{j=0}^{nd-1}(-1)^j\chi(j)q^{hj}(2j+1)^m.$$

Again, by (2.4), we have

$$q^{hnd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)q^{hx}(2x+1)^k d\mu_{-1}(x)$$
$$+\int_X \chi(x)q^{hx}(2x+1)^m d\mu_{-1}(x) = 2T_{m,\chi,q}^{(h)}(nd-1). \tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

**Theorem 2.1** *Let $n$ be odd positive integer. Then we obtain*

$$\frac{2\int_X \chi(x) q^{hx} e^{(2x+1)t} d\mu_{-1}(x)}{\int_X q^{hndx} e^{2ndtx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left(2T_{m,\chi,q}^{(h)}(nd-1)\right) \frac{t^m}{m!}.$$

Let $w_1$ and $w_2$ be odd positive integers. Then we set

$$S(w_1, w_2) = \frac{\int_X \int_X \chi(x_1)\chi(x_2) q^{h(w_1x_1+w_2x_2)} e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_X q^{hw_1w_2dx} e^{2w_1w_2dxt} d\mu_{-1}(x)}. \tag{2.8}$$

By Theorem 2.1 and (2.8), after calculations, we obtain

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2}\int_X \chi(x_1) q^{hw_1x_1} e^{(w_1(2x_1+1)+w_1w_2x)t} d\mu_{-1}(x_1)\right) \\ &\quad \times \left(\frac{2\int_X \chi(x_2) q^{hw_2x_2} e^{(2x_2+1)(w_2t)} d\mu_{-1}(x_2)}{\int_X q^{hw_1w_2dx} e^{2w_1w_2dtx} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} E_{m,\chi,q^{w_1}}^{(h)}(w_2x) w_1^m \frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} T_{m,\chi,q^{w_2}}^{(h)}(w_1d-1) w_2^m \frac{t^m}{m!}\right). \end{aligned} \tag{2.9}$$

By using Cauchy product in the above, we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} E_{j,\chi,q^{w_1}}^{(h)}(w_2x) w_1^j T_{m-j,\chi,q^{w_2}}^{(h)}(w_1d-1) w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.10}$$

From the symmetry of $S(w_1, w_2)$ in $w_1$ and $w_2$, we also see that

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2}\int_X \chi(x_2) q^{hw_2x_2} e^{(w_2(2x_2+1)+w_1w_2x)t} d\mu_{-1}(x_2)\right) \\ &\quad \times \left(\frac{2\int_X \chi(x_1) q^{hw_1x_1} e^{(2x_1+1)(w_1t)} d\mu_{-1}(x_1)}{\int_X q^{hw_1w_2dx} e^{2w_1w_2dtx} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} E_{m,\chi,q^{w_2}}^{(h)}(w_1x) w_2^m \frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} T_{m,\chi,q^{w_1}}^{(h)}(w_2d-1) w_1^m \frac{t^m}{m!}\right). \end{aligned}$$

Thus we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} E_{j,\chi,q^{w_2}}^{(h)}(w_1x) w_2^j T_{m-j,\chi,q^{w_1}}^{(h)}(w_2d-1) w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.11}$$

By comparing coefficients $\frac{t^m}{m!}$ on the both sides of (2.10) and (2.11), we arrive at the following theorem:

**Theorem 2.2** *Let $w_1$ and $w_2$ be odd positive integers. Then we obtain*

$$\begin{aligned}
&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^{j}E_{j,\chi,q^{w_2}}^{(h)}(w_1x)T_{m-j,\chi,q^{w_1}}^{(h)}(w_2d-1)\\
&=\sum_{j=0}^{m}\binom{m}{j}w_1^{j}w_2^{m-j}E_{j,\chi,q^{w_1}}^{(h)}(w_2x)T_{m-j,\chi,q^{w_2}}^{(h)}(w_1d-1),
\end{aligned}$$

*where $E_{k,\chi,q}^{(h)}(x)$ and $T_{m,\chi,q}^{(h)}(k)$ denote the second kind generalized $(h,q)$-Euler polynomials attached to $\chi$ and the alternating sums of powers of consecutive $(h,q)$-odd integers, respectively.*

By Theorem 2.2, we have the following corollary.

**Corollary 2.3** *Let $w_1$ and $w_2$ be odd positive integers. Then we obtain*

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j}x^{j-k}E_{k,\chi,q^{w_2}}^{(h)}T_{m-j,\chi,q^{w_1}}^{(h)}(w_2d-1)\\
&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j}w_2^{m-k}x^{j-k}E_{k,\chi,q^{w_1}}^{(h)}T_{m-j,\chi,q^{w_2}}^{(h)}(w_1d-1).
\end{aligned}$$

Now, we will derive another interesting identities for the second kind generalized $(h,q)$-Euler polynomials using the symmetric property of $S(w_1,w_2)$.

$$\begin{aligned}
S(w_1,w_2) &= \left(\frac{1}{2}\int_X \chi(x_1)q^{hw_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_{-1}(x_1)\right)\\
&\quad\times\left(\frac{2\int_X \chi(x_2)q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_X q^{hw_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right)\\
&= \left(\frac{1}{2}e^{w_1w_2xt}\int_X \chi(x_1)q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_{-1}(x_1)\right)\\
&\quad\times\left(2\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)q^{w_2hj}e^{(2j+1)(w_2t)}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)q^{w_2hj}E_{n,\chi,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

By using the symmetry property in (2.12), we also have

$$\begin{aligned} S(w_1, w_2) &= \left( \frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_2) q^{h w_2 x_2} e^{(2x_2+1) w_2 t} d\mu_{-1}(x_2) \right) \\ &\quad \times \left( \frac{2 \int_X \chi(x_1) q^{h w_1 x_1} e^{(2x_1+1)(w_1 t)} d\mu_{-1}(x_1)}{\int_X q^{h w_1 w_2 d x} e^{2 w_1 w_2 d t x} d\mu_{-1}(x)} \right) \\ &= \left( \frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_2) q^{h w_2 x_2} e^{(2x_2+1) w_2 t} d\mu_{-1}(x_2) \right) \\ &\quad \times \left( 2 \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} e^{(2j+1)(w_1 t)} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_2-1} (-1)^j \chi(j) q^{w_1 h j} E_{n,\chi,q^{w_2}}^{(h)} \left( w_1 x + (2j+1) \frac{w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

By comparing coefficients $\frac{t^n}{n!}$ on the both sides of (2.12) and (2.13), we have the following theorem.

**Theorem 2.4** *Let $w_1$ and $w_2$ be odd positive integers. Then we obtain*

$$\begin{aligned} &\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 h j} E_{n,\chi,q^{w_1}}^{(h)} \left( w_2 x + (2j+1) \frac{w_2}{w_1} \right) w_1^n \\ &= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} E_{n,\chi,q^{w_2}}^{(h)} \left( w_1 x + (2j+1) \frac{w_1}{w_2} \right) w_2^n. \end{aligned} \tag{2.14}$$

If we take $x = 0$ in Theorem 2.4, we also derive the interesting identity for the second kind generalized $(h, q)$-Euler numbers as follows:

$$\begin{aligned} &\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 h j} E_{n,\chi,q^{w_1}}^{(h)} \left( (2j+1) \frac{w_2}{w_1} \right) w_1^n \\ &= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 h j} E_{n,\chi,q^{w_2}}^{(h)} \left( (2j+1) \frac{w_1}{w_2} \right) w_2^n. \end{aligned}$$

Observe that if $q \to 1$, then (2.14) reduces to Theorem 2.4 in [6].

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