Encoding of Partition Set Using Sub-exceeding Function

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Abstract

We introduce in this paper a new method to encode all partitions of any set of \( n \) objects using sub-exceeding function. So, let \( n \) and \( k \) be two positive integers such that \( 1 \leq k \leq n \) and let too \( \Omega \) a set of cardinal \( n \). The study of the subset called \( G^k_n \) of \( F_n \) (set of sub-exceeding functions on \( [n] \)) leads us to the construction of a bijection between this set and the set of partitions of \( \Omega \) with \( k \)-subsets. Once the bijection is built, we can thus code all the partitions of the set \( \Omega \) by sub-exceeding functions and respectively to find an algorithm for the decoding.

Mathematics Subject Classification: 05A05; 05A18; 05A19; 18C35

Keywords: Sub-exceeding function, Statistical Permutation, Partition set

1 Introduction and Notation

Introduction

This paper presents the fruit of our results in the article entitled: "Part of a set and sub-exceeding function (Coding and Decoding) [4]" which presents, for a positive integer \( n \), the bijection between the subset \( \mathcal{H}_n \) of \( \mathcal{F}_n \) and the set \( \mathcal{P}(\Omega) \). Here \( \mathcal{F}_n \) represents the set of sub-exceeding functions on \( [n] \) and \( \mathcal{P}(\Omega) \) present all parts of the set \( \Omega \) where \( \text{Card}(\Omega) = n - 1 \).
The study of the sub-exceeding functions does not stop to present us surprising results and which we push to enrich our research. Let now $n$ and $k$ be two positive integers such that $1 \leq k \leq n$. From the bijection between the sets $\mathcal{H}_n$ and $\mathcal{P}(\Omega)$, we can encode any subset of $\Omega$ with one and only one sub-exceeding function in $\mathcal{H}_n$. So, from this result, a question arises if it is also possible to encode any partition of $\Omega$ which have $k$-subset by one and only one sub-exceeding function in $\mathcal{F}_n$. In terms of coding, this makes possible to speed up the processing time for any given information sequence. To give an answer for this question, we will analyze a set called $\mathcal{G}_n^k$ which is an extension of the set $\mathcal{H}_n^k$.

Our goal for this search will be to encode all partitions of $\Omega$ where Card($\Omega$) = $n$. That is to say, for any partition of $\Omega$ into $k$-subsets, there is one and only one sub-exceeding function in $\mathcal{G}_n^k$ who represents it, and conversely for a sub-exceeding function in $\mathcal{G}_n^k$, there exists one and only one corresponding subdivision of $\Omega$ into $k$ subsets.

**Notation**

Let us recall some notations that allow us to facilitate the reading of this paper.

1. $[n]$ : the set $\{1; 2; \ldots; n\}$,
2. $\mathbb{N}$ : the set of positive integers,
3. $\Omega$ : a set of $n$ elements such that $\Omega = \{\omega_1; \omega_2; \ldots; \omega_n\}$,
4. $\mathcal{P}_k(\Omega)$ : the set of the parts of $\Omega$ which have $k$ elements,
5. $\mathcal{P}_k(\Omega)$ : the set of partitions of $\Omega$ which have $k$ any subsets,

**2 Preliminaries**

**2.1 The set $\mathcal{F}_n$ and its properties**

**Definition 2.1.** Let $n$ be a positive integer such that $n > 0$ and let too $f$ a map from $[n]$ to $[n]$. This function $f$ is called sub-exceeding on $[n]$ if and only if $f(i) \leq i$ for all $i$ in $[n]$.

In the whole continuation we will represent $f$ by the word of $n$ letters $f(1)f(2)\ldots f(n)$. Thus we describe $f$ by his images i.e. $f = f(1)f(2)\ldots f(n)$. Moreover, we denote by $\mathcal{F}_n$ the set of this sub-exceeding function. So

$$\mathcal{F}_n = \{f : [n] \rightarrow [n] \mid f(i) \leq i, \forall i \in [n]\}.$$  (2.1)
Following this definition, the study of this set \( F_n \) presents us an important result introduced in the theorem as follow.

**Theorem 2.1** (Roberto Mantaci and Fanja Rakotondrajao [7]). Let \( \phi \) be the map from \( F_n \) to \( S_n \) defined by

\[
\phi : F_n \rightarrow S_n
\]

\[
f \mapsto \phi(f) = (n f(n))(n - 1 f(n - 1))...(2 f(2))(1 f(1)) = \sigma_f
\]

(2.2)

So, the map \( \phi \) is bijective

Here \((i f(i))\) is the permutation which transforms \( i \) into \( f(i) \) and \( f(i) \) into \( i \). Similarly \((i f(i))(j f(j))\) is the compose of two permutations \((i f(i))\) and \((j f(j))\). So

\[
\sigma_f = (n f(n))...(2 f(2))(1 f(1)) = (n f(n))\circ(n-1 f(n-1)\circ...\circ(1 f(1))
\]

(2.3)

**Example 2.2.** For \( n = 3 \) and \( f = 122 \), we have \( \phi(f) = (3 2)(2)(1) = 132 \) where the permutation \((i, i)\) is simplified by \((i)\).

### 2.2 The set \( H_n \) and its properties

**Definition 2.2.** For a positive integers \( n \) and \( k \) such that \( 1 \leq k \leq n \), we denote by \( H^k_n \) the sub-set of \( F_n \) such that

\[
H^k_n = \{ f \in F_n \mid f(i) \leq f(i + 1) \; \forall 1 \leq i \leq n - 1 \; \text{and} \; \text{Im}(f) = [k] \}.
\]

(2.4)

Here, \( H^k_n \) is the set of all sub-exceeding function of \( F_n \) with a quasi-increasing sequence of image formed by all elements of \([k]\).

**Example 2.3.** Take \( n = 4 \) and \( k = 3 \). We have here \( f = 1123 \in H^3_4 \) because \((f(i))_{i\in[4]}\) is an quasi-increasing sequence and all of the elements of \([3]\) are there. Now if we take \( f = 1133 \), although the sequence \((f(i))_{i\in[4]}\) is quasi-increasing, \( f = 1133 \notin H^3_4 \) because \( \text{Im}(f) \neq [3] \) (without 2 among the \( f(i) \)).

From this definition 2.2, we denote by \( H_n \) the set as

\[
H_n = \bigcup_{k=1}^{n} H^k_n.
\]

(2.5)

**Example 2.4.** Take \( n = 1, 2, 3 \)

\[
H^1_1 = 1 \quad H^2_1 = 11 \; \text{et} \; H^2_2 = 12 \quad H^3_3 = 111 , \; H^3_2 = 112, 122 \; \text{et} \; H^3_3 = 123
\]
Theorem 2.5. (Luc RABEFIHAVANANA [4]).
Let $n$ and $k$ be two integers such that $1 \leq k \leq n$.

1. For $k = 1$, we see always that $\mathcal{H}^1_n$ is the singleton set such that
   \[
   \mathcal{H}^1_n = \{ f = 111 \ldots 1_{n} \text{-fois} \}.
   \]

2. For $k = n$, we see also that $\mathcal{H}^n_n$ is the singleton set such that
   \[
   \mathcal{H}^n_n = \{ f = 123 \ldots (n-1)(n) \}.
   \]

3. For all integers $k$ such that $1 < k < n$, we can construct all sub-exceeding functions of $\mathcal{H}^k_n$ as follow:
   
   (a) Take all elements of $\mathcal{H}^{k-1}_{n-1}$ and add the integer $k$ at the end,
   
   (b) Take also all elements of $\mathcal{H}^k_{n-1}$ and add the integer $k$ at the end.

So, for the best presentation, we write this construction as
   \[
   \mathcal{H}^k_n = \{ \mathcal{H}^{k-1}_{n-1} \searrow k \} \cup \{ \mathcal{H}^k_{n-1} \searrow k \}.
   \]

Here, $(\ast) \searrow k$ means that we add the integer $k$ at the and of all elements of $(\ast)$.

Proposition 2.1. (Luc RABEFIHAVANANA [4]).
Let $n$ and $k$ be two integers such that $1 \leq k \leq n$. So, we have the following relations

1. Card $\mathcal{H}^k_n = $ Card $\mathcal{H}^{k-1}_{n-1} +$ Card $\mathcal{H}^k_{n-1}$

2. Card $\mathcal{H}^k_n = \binom{n-1}{k-1}$

3. Card $\mathcal{H}^k_n = 2^{n-1}$

From this proposition, we have the cardinal table of $\mathcal{H}^k_n$ as

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This table is a modified Pascal triangle. More precisely, it is shifted i.e. instead of \( k \in \mathbb{N} \), we have \( k \in \mathbb{N}^* \). Moreover, the values of the \( i^{th} \) line in this table is the value of \( (i-1)^{th} \) line of the Pascal triangle.

**Definition 2.3.** Let \( n \) and \( k \) be two integers such that \( 1 \leq k \leq n \). For a sub-exceeding function \( f \) in \( H_n^k \) and an integer \( i \) such that \( i \in [k] \), we define by \( \text{dpos}(i) \) the last position of \( i \) in \( f \).

**Example 2.6.** In \( H_4^4 \), take \( f = 111223334 \). We have here \( \text{dpos}(1) = 3 \), \( \text{dpos}(2) = 5 \) and \( \text{dpos}(3) = 8 \).

**Theorem 2.7.** (Luc RABEFIHAVANANA [4]).

Let \( n \) and \( k \) be two integers such that \( 1 \leq k \leq n \). Now let \( \psi \) be a map from \( H_n^k \) to \( P^{k-1}(\Omega) \) such that

\[
\psi : H_n^k \longrightarrow P^{k-1}(\Omega)
\]

\[
f \mapsto \psi(f) = \{\omega_{\text{dpos}(1)}; \omega_{\text{dpos}(2)}; \ldots; \omega_{\text{dpos}(k-1)}\}
\]

(2.6)

where

\[
\Omega = \{\omega_1; \omega_2; \ldots; \omega_{n-1}\}.
\]

(2.7)

Then, the mapping \( \psi \) is bijective.

From this theorem, \( H_n^k \) represent the set of the parts of \( \Omega \) to \( k-1 \) elements. Moreover, The map \( \psi \) is also bijective from \( H_n \) to \( P(\Omega) \).

As results, given an arbitrary subset of \( \Omega \), we can encode it by using a sub-exceeding function in \( H_n \). Reciprocally, if we have a sub-exceeding function in \( H_n \), we can decode it and find the subset of \( \Omega \) that it represents.

Now, as \( P(\Omega) \) is stable by the operations \( \bigcup \), \( \bigcap \) and by passing to the complement, so we present in this section the equivalent of these operations in \( H_n \).

**Definition 2.4.** Let \( f \) be a sub-exceeding function in \( H_n^k \), we define the indicator of \( f \) the set \( I_f \) such that

\[
I_f = \{\text{dpos}(1); \text{dpos}(2); \ldots; \text{dpos}(k-1)\}
\]

(2.8)

**Definition 2.5.** Let \( f \) and \( g \) be two sub-exceeding functions who belong respectively in \( H_n^k \) and \( H_n^l \). We denote by \( (f \cup_{\text{Rab}} g) \) the sub-exceeding function \( h \) such that

\[
h = (f \bigcup_{\text{Rab}} g) \in H_n^{m+1} \text{ avec } I_h = I_f \bigcup I_g.
\]

(2.9)

et \( m = \text{card } I_h \)
Example 2.8. Let \( f = 1122344 \in \mathcal{H}_7^3 \) and \( g = 1222345 \in \mathcal{H}_7^5 \) be two sub-exceeding functions. We have here \( I_f = \{2; 4; 5\} \) and \( I_g = \{1; 4; 5; 6\} \). Then, the sub-exceeding function \( h = (f \cup_{\text{Rab}} g) \) has his indicator the set \( I_h = \{1; 2; 4; 5; 6\} \). From this indicator, we have \( h \in \mathcal{H}_7^5 \) such that
\[
h = 1 2 3 3 4 5 6
\]

Theorem 2.9. (Luc RABEFIHAVANANA [4]).
Let \( f \) and \( g \) be two sub-exceeding functions in \( \mathcal{H}_n \) which respectively represents \( \Omega_f \) and \( \Omega_g \) (two subset of \( \Omega \)). Then, \( \mathcal{H}_n \) is stable by \( \cup_{\text{Rab}} \) and \( (f \cup_{\text{Rab}} g) \) represents \( \Omega_f \cup \Omega_g \).

Definition 2.6. Let \( f \) and \( g \) be two sub-exceeding function who belong respectively in \( \mathcal{H}_n^k \) and \( \mathcal{H}_n^l \). We denote by \( (f \cap_{\text{Rab}} g) \) the sub-exceeding function \( h \) such that
\[
h = (f \cap_{\text{Rab}} g) \in \mathcal{H}_n^{m+1}
\]
avec \( I_h = I_f \cap I_g \).

Example 2.10. Let \( f = 1122344 \in \mathcal{H}_7^3 \) and \( g = 1222345 \in \mathcal{H}_7^5 \) be two sub-exceeding functions. We have here \( I_f = \{2; 4; 5\} \) and \( I_g = \{1; 4; 5; 6\} \). Then, the sub-exceeding function \( h = (f \cap_{\text{Rab}} g) \) has his indicator the set \( I_h = \{4; 5\} \). From this indicator, we have \( h \in \mathcal{H}_7^3 \) such that
\[
h = 1 1 1 1 2 3 3
\]

Theorem 2.11. (Luc RABEFIHAVANANA [4]).
Let \( f \) and \( g \) be two sub-exceeding functions in \( \mathcal{H}_n \) which respectively represents \( \Omega_f \) and \( \Omega_g \) (two subset of \( \Omega \)). Then, \( \mathcal{H}_n \) is stable by \( \cap_{\text{Rab}} \) and \( (f \cap_{\text{Rab}} g) \) represent \( \Omega_f \cap \Omega_g \).

Definition 2.7. Let \( f \) be the sub-exceeding function defined on \( \mathcal{H}_n^k \). We denote by \( \bar{f} \) the complementary sub-exceeding function of \( f \) such that \( \bar{f} \in \mathcal{H}_n^{n-k+1} \) with indicator \( I_f = \bar{I}_f \). Here, \( \bar{I}_f \) is the complementary set of \( I_f \) in \([n-1]\).

Example 2.12. Let \( f = 1122344 \in \mathcal{H}_7^3 \). we have \( I_f = \{2; 4; 5\} \) and \( I_f = \{1; 3; 6\} \). The complement sub-exceeding function of \( f \) denoted by \( \bar{f} \) is then
\[
\bar{f} = 1 2 2 3 3 3 4 \in \mathcal{H}_7^4
\]

Theorem 2.13. (Luc RABEFIHAVANANA [4]).
Let \( f \) be a sub-exceeding function in \( \mathcal{H}_n \) which represent \( \Omega_f \). So, \( \bar{f} \) represents \( \Omega_f \) the complement of \( \Omega_f \) in \( \Omega \).
3 Main result: bijection between the set $G_n^k$ and $P_k(\Omega)$

3.1 The set $G_n^k$ and its properties

Definition 3.1. Let $n$ be a not zero positive integer. A sub-exceeding function $f$ on $[n]$ satisfies the condition denoted $(E)$ if for any integer $m$ in $[n]$ such that $m > 1$ and $m \in \{f(i)\}_{i \in [n]}$, by posing $m = f(j)$, there is another integer $j'$ where $j' < j$ such that $f(j') = m - 1$.

In another way, a sub-exceeding function $f$ on $[n]$ satisfies the condition $(E)$ if and only if for any integer $m$ ($m > 1$) in the sequence of image $(f(i))_{i \in [n]}$, all integers from 1 to $m - 1$ must be found before $m$ in this sequence.

Example 3.1. In $F_7$, let’s take two sub-exceeding functions $f$ and $g$ such that $f = 1123124$ and $g = 1123125$. Here the function $f$ satisfies the condition $(E)$ but $g$ is not. Taking $g(7)$ which is equal to 5 but all integers 1 to 4 are not found before this value (the integer 4 is not found in the sequence 112312).

Definition 3.2. Let $n$ and $k$ be two positive integers such as $1 \leq k \leq n$. We define by $G_n^k$ the subset of $F_n$:

$$G_n^k = \left\{ f \in F_n \text{ such that } \{f(i)\}_{i \in [n]} = [k] \text{ and } f \text{ satisfies (E)} \right\}.$$

From this definition, $G_n^k$ groups all sub-exceeding functions of $F_n$ having as image all the elements of $[k]$ and which checks the condition $(E)$.

Example 3.2. Take $f = 1213 \in G_4^3$ a sub-exceeding function. We have $f \in G_4^3$ because $\{f(i)\}_{i \in [4]} = [3]$ and that $f$ check the condition $(E)$. Now, if we take $g = 1132$, even if $\{f(i)\}_{i \in [4]} = [3]$, we have $g = 1132 \notin G_4^3$ because the condition $(E)$ is not checked (no 2 before 3).

Corollary 3.3. Let $n$ and $k$ be two positive integers such that $1 \leq k \leq n$. We have the relation

$$H_n^k \subseteq G_n^k \quad (3.1)$$

Proof. Since $H_n^k$ groups all sub-exceeding function of $F_n$ which have a quasi-increasing image formed by all elements of $[k]$, that is to say

$$H_n^k = \left\{ f \in F_n \mid f(i) \leq f(i+1) \text{ for all } i \in [n] \text{ such that } \{f(i)\}_{i \in [n]} = [k] \right\}.$$

From this definition of $H_n^k$ and from the condition $f(i) \leq f(i+1)$ for all $i \in [n]$, all elements of $H_n^k$ satisfy the condition $(E)$. So, we find that

$$H_n^k \subseteq G_n^k.$$
3.2 Iterative construction of $\mathcal{G}_n^k$

Example 3.4. To initialize the construction of the set $\mathcal{G}_n^k$, let’s start with $n = 1$ and $n = 2$. So, we have

\[ \mathcal{G}_1^1 = \{ 1 \} \]
\[ \mathcal{G}_2^1 = \{ 11 \} \text{ and } \mathcal{G}_2^2 = \{ 12 \} \]

Theorem 3.5. Let $n$ and $k$ be two integers such that $1 \leq k \leq n$.

1. For $k = 1$, we always see that $\mathcal{G}_n^1$ is the singleton set such as
   \[ \mathcal{G}_n^1 = \{ f \mid f = 111 \ldots 1_{n \text{-fois}} \} \]

2. For $k = n$, we see also that $\mathcal{G}_n^n$ is the singleton set such as
   \[ \mathcal{G}_n^n = \{ f \mid f = 123 \ldots (n - 1)(n) \} \]

3. For all integer $k$ such that $1 < k < n$, we can construct all elements of $\mathcal{G}_n^k$ as follow:
   
   (a) Take all elements of $\mathcal{G}_{n-1}^{k-1}$ and add the integer $k$ at the end,
   
   (b) Take also all elements of $\mathcal{G}_{n-1}^k$ and add all integer in $[k]$ one by one at the end.

So, for the best presentation, we write this construction as

\[ \mathcal{G}_n^k = \{ \mathcal{G}_{n-1}^{k-1} \cup k \} \bigcup \left\{ \mathcal{G}_{n-1}^k \cup \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} \right\} . \quad (3.2) \]

Here $(\ast) \cup k$ (resp. $(\ast) \cup \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix}$) means that we add the integer $k$

(resp. add one by one all elements of $[k]$) at the end of all elements of $(\ast)$.

Proof. 1. For $k = 1$, by definition of $\mathcal{G}_n^k$, we have

\[ \mathcal{G}_n^1 = \left\{ f \in \mathcal{F}_n \mid \{ f(i) \}_{i \in [n]} = [1] \text{ and that } f \text{ satisfies } (\mathcal{E}) \right\} , \]

i.e. $f(i) = 1$ for all $i \in [n]$. Then,

\[ \mathcal{G}_n^1 = \{ f = 111 \ldots 1_{n\text{-fois}} \} . \]
2. For \( k = n \), still by the definition of \( \mathcal{G}^k_n \), we have

\[
\mathcal{G}^n_n = \left\{ f \in \mathcal{F}_n \mid \{ f(i) \}_{i \in [n]} = [n] \text{ and that } f \text{ satisfie } (E) \right\}.
\]

However, by definition of a sub-exceeding function on \([n]\), it is only the function \( f = 123...n \) which is the sub-exceeding function that has an image sequence \( \{ f(i) \}_{i \in [n]} \) that is equal to the set \([n]\).

\[
\mathcal{G}^n_n = \{ f = 123...(n-1)(n) \}.
\]

3. For all integer \( k \) such that \( 1 < k < n \), prove by recurrence on \( n \) that

\[
\mathcal{G}^k_n = \{ \mathcal{G}^{k-1}_{n-1} \Join k \} \cup \left\{ \mathcal{G}^k_{n-1} \Join \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} \right\}.
\] (3.3)

- At order 2, our statement 3.3 is true (see example 3.4) but we still try to see for the order 3. Let’s prove that \( \mathcal{G}^2_3 = \{ \mathcal{G}^1_2 \Join 2 \} \cup \left\{ \mathcal{G}^2_2 \Join \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \).

First, we find that

\[
\mathcal{G}^k_3 = \begin{cases} 111 & \text{for } k = 1 \\ 112, 121, 122 & \text{for } k = 2 \\ 123 & \text{for } k = 3 \end{cases}
\]

As \( \mathcal{G}^1_2 = 11 \) and that \( \mathcal{G}^2_2 = 12 \), directly we find that the set \( \mathcal{G}^2_3 \) are formed by:

(a) the element of \( \mathcal{G}^1_2 \) by adding integer 2 at the end
(b) the element of \( \mathcal{G}^2_2 \) by adding one by one the integers 1 and 2 at the end.

- Suppose that the statement 3.3 is true up to the order \( n - 1 \) that is to say for any integer \( k \) such that \( 1 < k < n - 1 \), we have

\[
\mathcal{G}^k_{n-1} = \{ \mathcal{G}^{k-1}_{n-2} \Join k \} \cup \left\{ \mathcal{G}^k_{n-2} \Join \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} \right\}.
\]

- Let’s show now that our proposition is still true to the order \( n \). So, let \( k \) be an integer such that \( 1 < k < n \),
(a) First, let $f$ be a sub-exceeding function in $G_{n-1}^{k-1}$, that is to say that $f$ satisfies the condition (E) and that $\{f(i)\}_{i \in [n-1]} = [k-1]$. By adding the integer $k$ to the end of $f$, we have a new sub-exceeding function $f'$ on $[n]$ such that

$$f' = f(1)f(2)...f(n-1)k \text{ with } k = f(n).$$

Directly we find that $\{f(i)\}_{i \in [n]} = [k]$ and that the condition (E) is still checked.

(b) Next, let $f$ be a sub-exceeding function in $G_{n-1}^k$ that is, $f$ satisfies the condition (E) and that $\{f(i)\}_{i \in [n-1]} = [k]$. By adding an element of the integers $1, 2, 3, ..., k$ at the end of $f$, we have a new sub-exceeding function on $[n]$ denoted $f'$ such that

$$f' = f(1)f(2)...f(n-1)j \text{ with } j = f(n) \text{ where } j \in [k].$$

Directly we find that $\{f(i)\}_{i \in [n]}$ still remains equal to $[k]$ and that the condition (E) remains checked.

Then in both cases (a) and (b), the new function under $f'$ is an element of $G_n^k$. So

$$\{G_{n-1}^{k-1} \cap k\} \cup \left\{G_{n-1}^k \setminus \left( \frac{1}{2} \vdots \frac{k}{k} \right) \right\} \subseteq G_n^k \quad (3.4)$$

Conversely, let $f$ be a subexceeding function in $G_n^k$, we have $\{f(i)\}_{i \in [n]} = [k]$ and that $f$ also checks the condition (E). By removing $f(n)$ from $\{f(i)\}_{i \in [n]}$, we have a new sub-exceeding function $f'$ on $[n-1]$ such that

$$f' = f(1)f(2)...f(n-1).$$

Two cases occur, either we have the equality $\{f(i)\}_{i \in [n-1]} = [k-1]$ or we have $\{f(i)\}_{i \in [n-1]} = [k]$. So, in both cases, the set $\{f(i)\}_{i \in [n-1]}$ checks again (E). So

$$f' \in G_{n-1}^{k-1} \bigcup G_{n-1}^k.$$

consequently,

$$\{G_{n-1}^{k-1} \cap k\} \cup \left\{G_{n-1}^k \setminus \left( \frac{1}{2} \vdots \frac{k}{k} \right) \right\} \supseteq G_n^k \quad (3.5)$$
In sum, by the relation (3.4) and (3.5)

\[ \left\{ G_{n-1}^{k-1} \cup k \right\} \bigcup \left\{ G_{n-1}^k \cup \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} \right\} = G_n^k \]

Example 3.6. Here is a table which represents the set \( G_n^k \) for \( n = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
<th>( n ) ( \backslash ) ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>111</td>
<td>112</td>
<td>121</td>
<td>123</td>
<td>122</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1112</td>
<td>1123</td>
<td>1211</td>
<td>1213</td>
<td>1234</td>
<td>1222</td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Corollary 3.7. Let \( n \) and \( k \) be two integers such that \( 1 \leq k \leq n \).

1. For \( k = 1 \) and \( k = n \), we have \( \text{Card} \ G_n^k = 1 \).

2. For \( 1 < k < n \), we have

\[ \text{Card} \ G_n^k = \text{Card} \ G_{n-1}^{k-1} + k.\text{Card} \ G_{n-1}^k. \quad (3.6) \]

Proof. From the construction of the set \( G_n^k \) above (in the theorem (3.5)), we have these relations.

Corollary 3.8. The cardinal table of \( G_n^k \) is as follows based on \( n \) and \( k \).

<table>
<thead>
<tr>
<th>( n ) ( \backslash ) ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>15</td>
<td>25</td>
<td>10</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>\vdots</td>
</tr>
</tbody>
</table>
Proof. As $G_1^1 = 1$, $G_2^1 = 11$ et que $G_2^2 = 12$, by combining with the equation (3.6) which says that $\text{Card } G_n^k = \text{Card } G_{n-1}^{k-1} + k.\text{Card } G_{n-1}^k$, we have this table directly.

Looking at this table and the equation (3.6), we find that our cardinal table for $G_n^k$ is none other than the second-type Stirling table. So by choosing two not-zero positive integers $n$ and $k$ where $1 \leq k \leq n$, the integer obtained on this table represents the possible number of cases for the partition of any set of $n$ elements into $k$ subsets.

### 3.3 Bijection between $G_n^k$ and $\mathcal{P}_k(\Omega)$

**Notation 3.9.** Let $f$ be a sub-exceeding function in $G_n^k$. For all $i \in [k]$, we denote by $A_i$ the set of antecedents of $i$ per $f$. That is to say

$$A_i = f^{-1}(i).$$

**Example 3.10.** Let $f$ an element of $G_9^4$ such that $f = 111223214$. So, $A_1 = \{1, 2, 3, 8\}$, $A_2 = \{4, 5, 7\}$, $A_3 = \{6\}$ and $A_4 = \{9\}$.

**Proposition 3.1.** Let $f$ be a sub-exceeding function in $G_n^k$ represented by these sets of antecedents $\{A_1, A_2, ..., A_k\}$. Let now $i$ an integer in $[k]$, we find that the first integer $i$ of $f$ is placed at the $j^\text{ème}$ place where $j = \text{Min} A_i$.

**Proof.** Let $f$ be a sub-exceeding function in $G_n^k$ represented by these sets of antecedents $\{A_1, A_2, ..., A_k\}$. Let now $i$ an integer in $[k]$, as $A_i = f^{-1}(i)$, so $A_i$ groups all the positions of $i$ in $f$. Consequently the first integer $i$ in $f$ is at the $j^\text{ème}$ position where $j = \text{Min} A_i$.

**Example 3.11.** Let $f = 111223214$ an element of $G_9^4$ with $A_1 = \{1, 2, 3, 8\}$, $A_2 = \{4, 5, 7\}$, $A_3 = \{6\}$ and $A_4 = \{9\}$.

Here we easily find that the first integer 2 of $f$ is placed at 4th position which is none other than Min$A_2$ and this is the even for other integers in [4].

**Theorem 3.12.** Let $\varphi$ be the map from $G_n^k$ to $\mathcal{P}_k(\Omega)$ defined by

$$\varphi : G_n^k \rightarrow \mathcal{P}_k(\Omega), \quad f \mapsto \varphi(f) = \left\{ \{\omega_i\}_{i \in A_1}, \{\omega_i\}_{i \in A_2}, ..., \{\omega_i\}_{i \in A_k} \right\},$$

(3.7)

So, the map $\varphi$ is bijective.

**Proof.** As we have seen above that the cardinal of $G_n^k$ and $\mathcal{P}_k(\Omega)$ are the same and that it is the value of second-type Stirling table at the $n^{th}$ line and $k^{th}$ column. We have to show that the map $\varphi$ is injective.
Let $f$ and $g$ be two sub-exceeding functions in $G_n^k$ such that $\varphi(f) = \varphi(g)$. That is to say
\[
\left\{ \{\omega_i\}_{i \in A_1(f)}, \{\omega_i\}_{i \in A_2(f)}, \ldots, \{\omega_i\}_{i \in A_k(f)} \right\} = \left\{ \{\omega_i\}_{i \in A_1(g)}, \{\omega_i\}_{i \in A_2(g)}, \ldots, \{\omega_i\}_{i \in A_k(g)} \right\}.
\]
(3.8)
By the definition of a sub-exceeding function on $[n]$, we have always $f(1) = 1$ (resp $g(1) = 1$). As $\cap_{j \in [k]} \left( \{\omega_i\}_{i \in A_j(f)} \right) = \emptyset$ and $\cap_{j \in [k]} \left( \{\omega_i\}_{i \in A_j(g)} \right) = \emptyset$, we have necessary
\[
\left\{ \{\omega_i\}_{i \in A_1(f)} = \{\omega_i\}_{i \in A_1(g)} \right\}.
\]
(3.9)
Let now $p$ be the first integer of $[k]$ such that $\{\omega_i\}_{i \in A_p(f)} \neq \{\omega_i\}_{i \in A_p(g)}$ i.e. for all integer $j$ such that $j < p$, we have
\[
\{\omega_i\}_{i \in A_j(f)} = \{\omega_i\}_{i \in A_j(g)} \quad \text{but} \quad \{\omega_i\}_{i \in A_p(f)} \neq \{\omega_i\}_{i \in A_p(g)}.
\]
(3.10)
From the equality in (3.8), there exists an integer $p'$ such that $p' > p$ and
\[
\{\omega_i\}_{i \in A_p(f)} = \{\omega_i\}_{i \in A_{p'}(g)}
\]
(3.11)
Note by $m$ the integer such that $m = \text{Min} A_p$. So the first integer $p$ in the sequence of image of $f$ is placed at the $m^{th}$ position. By definition of a sub-exceeding function in $G_n^k$, all integers in $[p-1]$ are found before $p$. Consequently, we can write
\[
f = f(1) f(2) \ldots f(m-1) p f(m+1) \ldots f(n) \quad \text{where} \quad \{f(i)\}_{i \in [m-1]} = [p-1].
\]
(3.12)
As $\{\omega_i\}_{i \in A_p(f)} = \{\omega_i\}_{i \in A_p(g)}$, we have $A_p(f) = A_p(g)$. Consequently, we have also the equality $m = \text{Min} A_{p'}$. So, the first integer $p'$ of $g$ is placed also at the $m^{th}$ position.
As $\{\omega_i\}_{i \in A_j(f)} = \{\omega_i\}_{i \in A_j(g)}$ for all integer $j$ such that $j < p$, so $f(i) = g(i)$ for all integer $i$ such that $i < m$. Then, from the relation (3.12) which say that $\{f(i)\}_{i \in [m-1]} = [p-1]$, we can write $\{g(i)\}_{i \in [m-1]} = [p-1]$. Consequently,
\[
g = g(1) g(2) \ldots g(m-1) p' g(m+1) \ldots g(n) \quad \text{où} \quad \{g(i)\}_{i \in [m-1]} = [p-1].
\]
(3.13)
Or the integer $p'$ is strictly more than $p$, so it is not possible if $g$ is an element of $G_n^k$ because the integer $p$ is not found before $p'$ in the sequence of image $\{g(i)\}_{i \in [n]}$.

Necessary, if $f$ and $g$ are two sub-exceeding functions in $G_n^k$ such that $\varphi(f) = \varphi(g)$, we have always the equality $\{\omega_i\}_{i \in A_j(f)} = \{\omega_i\}_{i \in A_j(g)}$ for all integer $j$ such that $j \leq k$. Finally, we have
\[
f = g
\]
and the map $\varphi$ is injective. In sum $\varphi$ is bijective. \qed
3.4 Algorithm for the construction of the sub-exceeding function $f$ corresponding of any partition in $P_k(\Omega)$

Let $n$ and $k$ be two integers such that $1 \leq k \leq n$. Let also $\Omega$ an any set of cardinality $n$ such that

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}.$$ 

Take now $S = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ a partition of $\Omega$ into $k$ subsets. To construct the corresponding sub-exceeding function $f$ of $S$ in $G_n^k$, we have necessary three steps:

1. Search the set $J_{\Omega_i}$ which is the set of all position of the elements of $\Omega_i$ in $\Omega$.

2. We must order $S$ as we have $S = \{\Omega_1', \Omega_2', \ldots, \Omega_k'\}$ where

$$\text{Min } J_{\Omega_1'} < \text{Min } J_{\Omega_2'} < \ldots < \text{Min } J_{\Omega_k'}. \quad (3.14)$$

3. Finally, we have

$$J_{\Omega_1'} = f^{-1}(1)$$
$$J_{\Omega_2'} = f^{-1}(2)$$
$$\vdots$$
$$J_{\Omega_k'} = f^{-1}(k) \quad (3.15)$$

**Example 3.13.** Let $\Omega$ be the set such that $\Omega = \{a, b, c, d, e, f, g, h\}$ and $S = \{\{h\}; \{b, c\}; \{a, e, g\}; \{d, f\}\}$. Here we have

$$\Omega_1 = \{h\}$$
$$\Omega_2 = \{b, c\}$$
$$\Omega_3 = \{a, e, g\}$$
$$\Omega_4 = \{d, f\}$$

Now, let's present the three step to construct the corresponding sub-exceeding $f$ which is an element of $G_8^4$.

1. First, we have

$$J_{\Omega_1} = \{8\}$$
$$J_{\Omega_2} = \{2, 3\}$$
$$J_{\Omega_3} = \{1, 5, 7\}$$
$$J_{\Omega_4} = \{4, 6\}$$

and that

$$\text{Min } J_{\Omega_1} = 8$$
$$\text{Min } J_{\Omega_2} = 2$$
$$\text{Min } J_{\Omega_3} = 1$$
$$\text{Min } J_{\Omega_4} = 4.$$
2. Secondly, after calculating $\text{Min} J_A$ above for all $i$, we can write $S = \{\Omega'_1, \Omega'_2, \Omega'_3, \Omega'_4\}$ such that

$$
\Omega'_1 = \Omega_3 = \{1, 5, 7\}
$$
$$
\Omega'_2 = \Omega_2 = \{2, 3\}
$$
$$
\Omega'_3 = \Omega_4 = \{4, 6\}
$$
$$
\Omega'_4 = \Omega_1 = \{8\}.
$$

3. Finally, we found that

$$
J_{\Omega'_1} = f^{-1}(1) = \{1, 5, 7\}
$$
$$
J_{\Omega'_2} = f^{-1}(2) = \{2, 3\}
$$
$$
J_{\Omega'_3} = f^{-1}(3) = \{4, 6\}
$$
$$
J_{\Omega'_4} = f^{-1}(4) = \{8\}.
$$

Consequently

$$f = 12231314.$$

Acknowledgements. The authors would like to thank the anonymous referees for their careful reading of our manuscript and helpful suggestions.

Ethics. The authors declare that there is no conflict interests regarding the publication of this manuscript.

References


Received: January 21, 2018; Published: March 20, 2018