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A New Generalization of Weibull Distribution

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Abstract

The Lomax-Weibull distribution is introduced as a new generalization of Weibull distribution. Some properties of the new distribution are investigated. Also, maximum likelihood and Bayesian estimation of the four unknown parameters are discussed. The asymptotic variance-covariance matrix is obtained. Finally, a numerical example is provided.

Keywords: The Lomax-Weibull (Lomax-W) distribution, maximum likelihood estimates, asymptotic variance-covariance matrix, Bayesian estimation

1 Introduction

Parametric modeling of data is in the main stream of statistical inference, and the Weibull distribution had been widely used for modeling many types of data, and reliability data in particular. As it has been noted and suggested by many authors, the Weibull distribution as such may not give adequate fits in many data. For this reason, among many others, many authors suggested new families of distributions in the hope of adding flexibility in the modeling process. Many of these families are modification, extension or combinations of existing one. Among these new-families: "the beta-G family (Eugene et al.,2002; Jones,2004); the gamma-G family(type 1)(Zografos and Balakrishanan, 2009); the Kumaraswamy-G (Kw-G)family(Cordeiro and de Castro,2011); the McDonald - G (Mc-G) family (Alexander *et al.*, 2012); exponentiated generalized-G family (Cordeiro *et al.*, 2013); the transformed -

transformer T-X family (Alzaatreh *et al.*, 2013); the exponentiated T-X family (Alzaghal *et al.*, 2013); the Lomax-*Fréchet* and the Lomax-Gumbel distributions(Gupta et al., 2015 and Gupta et al., 2016, respectively)"and more. In the following section we will concern with the last one.

Gupta et al., 2015 and Gupta *et al.* (2016) suggested a new obtaining family of distributions from two other families F and G with the following cdf:

$$F_G(x) = \frac{F(G(x))}{F(1)} \tag{1}$$

and the corresponding pdf is

$$f_G(x) = \frac{F'(G(x))g(x)}{F(1)} \tag{2}$$

where F(x) and f(x) be the cdf and pdf of the first distribution with support $[0, a], 1 \leq a < \infty$ and G(x) and g(x) be the cdf and pdf of the second distribution with support $[a, b]$, where $b, c \in R$ or $[0, \infty) or (-\infty, \infty)$. For different choices of first and second distributions one can construct a large number of distributions such as Gupta et al., 2015 introduced the Lomax-*Fréchet* distribution; Gupta et al., 2016 introduced the Lomax-Gumbel distribution.

2 Lomax-Weibull Distribution

The cdf for Lomax distribution is $F(x) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}$ and the cdf of Weibull distribution is $G(x) = 1 - e^{-\beta x^{\gamma}}$. It is suggested here to use the idea of Gupta aem et al. (2015)and Gupta *et al.* (2016) by putting F(x) and G(x) in formulas (1) and (2), to generate what may be terms of the Lomax-W distribution.

The cdf of the Lomax-W distribution is

$$F_{LW}(x) = K[1 - (1 + \frac{1}{\lambda}(1 - e^{-\beta x^{\gamma}}))^{-\alpha}], \tag{3}$$

and

$$f_{LW}(x) = \frac{K\alpha\beta\gamma}{\lambda} x^{\gamma-1} e^{-\beta x^{\gamma}} (1 + \frac{1}{\lambda}(1 - e^{-\beta x^{\gamma}}))^{-(\alpha+1)} \tag{4}$$

where the constant $K = [F(1)]^{-1} = [1 - (1 + \frac{1}{\lambda})^{-\alpha}]^{-1}$, and $\alpha, \lambda, \beta, \gamma > 0, x \geq 0$. using (3) and (4), one can obtain the following hazard function:

$$h_{LW}(x) = \frac{K\alpha\beta\gamma}{\lambda} \frac{x^{\gamma-1} e^{-\beta x^{\gamma}} (1 + \frac{1}{\lambda}(1 - e^{-\beta x^{\gamma}}))^{-(\alpha+1)}}{1 - K[1 - (1 + \frac{1}{\lambda}(1 - e^{-\beta x^{\gamma}}))^{-\alpha}]} \tag{5}$$

For different parameter values, figures 1-2 show plots of the pdf and hazard function of the Lomax-W distribution as follows:.

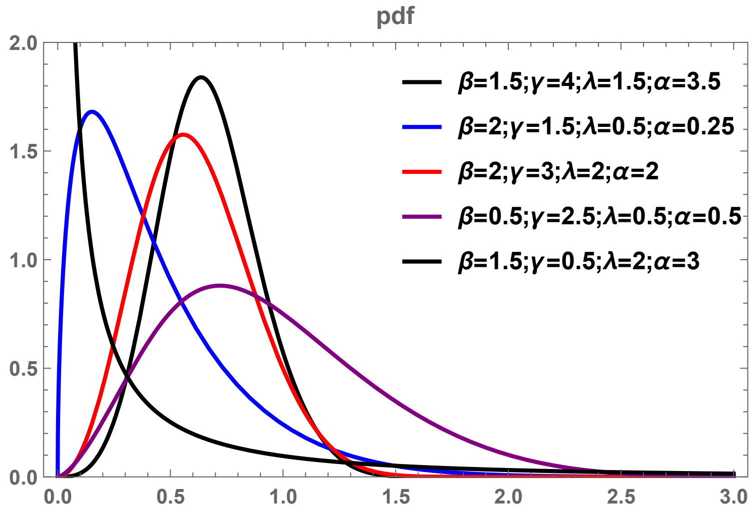

Figure 1: pdf of Lomax-W distribution for various values of the parameters

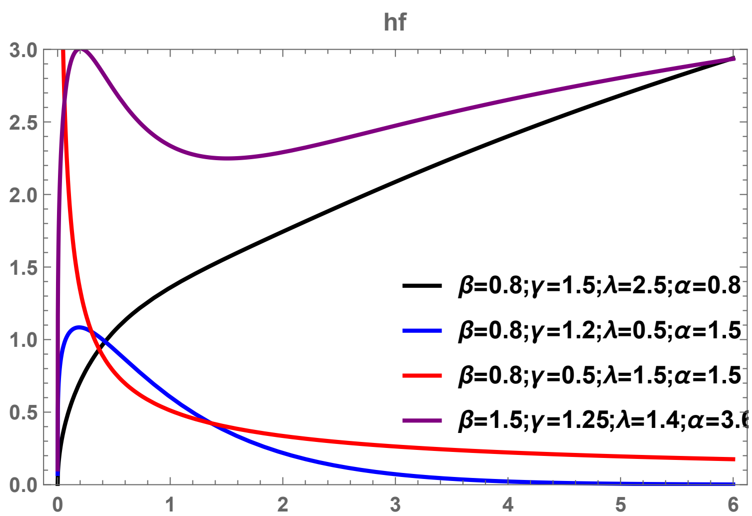

Figure 2: Hazard function of Lomax-W distribution for various values of the parameters

The quantile function of the Lomax-W distribution is given by:

$$x_q = [-\frac{1}{\beta} log[1 - \lambda((1 - \frac{q}{K})^{-\frac{1}{\alpha}} - 1)]]^{\frac{1}{\gamma}}. \tag{6}$$

This formula can be used to generate random numbers from the Lomax-W distribution and calculate some expressions such as skewness, kurtosis, L-moments and others.

The r^{th} raw moments for the Lomax-W distribution is

$$E(X^r) = \frac{K\alpha\beta}{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\lambda^{-i}C_i^{-(\alpha+1)}C_j^i(\beta(j+1))^{-(\frac{r}{\gamma}+1)}\Gamma(\frac{r}{\gamma}+1) \quad (7)$$

where $C_j^i = \frac{\Gamma(i+1)}{\Gamma(i-j+1)\Gamma(j+1)}$.

3 Parameter Estimation

Now, the maximum likelihood and Bayesian Estimation are discussed.

3.1 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n follows the Lomax-W distribution. To estimate the four unknown parameters $(\alpha, \lambda, \beta, \gamma)$ of the Lomax-W distribution we first obtain the likelihood function as follows:

$$L = (\frac{K\alpha\beta\gamma}{\lambda})^n\prod_{i=1}^{n}x_i^{\gamma-1}e^{-\beta\sum_{i=1}^{n}x_i^{\gamma}}\prod_{i=1}^{n}(1+\frac{1}{\lambda}(1-e^{-\beta x_i^{\gamma}}))^{-(\alpha+1)}, \quad (8)$$

and the log-likelihood function is

$$lnL = nlnK + nln\alpha + nln\beta + nln\gamma - nln\lambda + (\gamma-1)\sum_{i=1}^{n}lnx_i - \beta\sum_{i=1}^{n}x_i^{\gamma} - (\alpha+1)\sum_{i=1}^{n}ln(1+\frac{1}{\lambda}(1-e^{-\beta x_i^{\gamma}})). \quad (9)$$

The first order derivative of the log-likelihood function with respect to α, λ, β and γ are

$$\begin{aligned}
\frac{\partial lnL}{\partial\alpha} &= -\frac{n(1+\lambda^{-1})^{-\alpha}ln(1+\lambda^{-1})}{(1-(1+\frac{1}{\lambda})^{-\alpha})} + \frac{n}{\alpha} - \sum_{i=1}^{n}ln(1+\frac{1}{\lambda}(1-e^{-\beta x_i^{\gamma}})),\\
\frac{\partial lnL}{\partial\lambda} &= \frac{n\alpha(1+\lambda^{-1})^{-(\alpha+1)}}{\lambda^2(1-(1+\frac{1}{\lambda})^{-\alpha})} - \frac{n}{\lambda} + \frac{(\alpha+1)(1-e^{-\beta x_i^{\gamma}})}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^{\gamma}}))},\\
\frac{\partial lnL}{\partial\beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}x_i^{\gamma} - \frac{(\alpha+1)}{\lambda}\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{-\beta x_i^{\gamma}}}{(1+\lambda^{-1}(1-e^{-\beta x_i^{\gamma}})},\\
\frac{\partial lnL}{\partial\gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n}lnx_i - \beta\sum_{i=1}^{n}x_i^{\gamma}lnx_i - \frac{\beta(\alpha+1)}{\lambda}\sum_{i=1}^{n}\frac{e^{-\beta x_i^{\gamma}}x_i^{\gamma}lnx_i}{(1+\lambda^{-1}(1-e^{-\beta x_i^{\gamma}}))}.
\end{aligned} \quad (10)$$

To obtain the estimates of the four unknown parameters, one can equate formulas in (10) to zero and solving them numerically. The second order derivative of log-likelihood function are:

$$I_{\alpha,\alpha}=\frac{\partial^2 lnL}{\partial\alpha^2} = -\frac{n}{\alpha^2}+\frac{n(1+\lambda^{-1})^{-\alpha}(ln(1+\lambda^{-1}))^2}{[1-(1+\lambda^{-1})^{-\alpha}]}+\frac{n(1+\lambda^{-1})^{-2\alpha}(ln(1+\lambda^{-1}))^2}{[1-(1+\lambda^{-1})^{-\alpha}]^2},$$

$$I_{\alpha,\lambda}=\frac{\partial^2 lnL}{\partial\alpha\partial\lambda} = \frac{n(1+\lambda^{-1})^{-(\alpha+1)}[1-\alpha ln(1+\lambda^{-1})]}{\lambda^2[1-(1+\lambda^{-1})^{-\alpha}]}+\frac{n\alpha(1+\lambda^{-1})^{-(2\alpha+1)}ln(1+\lambda^{-1})}{\lambda^2[1-(1+\lambda^{-1})^{-\alpha}]^2}+\sum_{i=1}^{n}\frac{(1-e^{-\beta x_i^\gamma})}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))}$$

$$I_{\alpha,\beta}=\frac{\partial^2 lnL}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n}\frac{x_i^\gamma e^{-\beta x_i^\gamma}}{\lambda(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))},$$

$$I_{\alpha,\gamma}=\frac{\partial^2 lnL}{\partial\alpha\partial\gamma} = -\sum_{i=1}^{n}\frac{\beta x_i^\gamma e^{-\beta x_i^\gamma} lnx_i}{\lambda(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))},$$

$$I_{\lambda,\lambda}=\frac{\partial^2 lnL}{\partial\lambda^2} = \frac{n}{\lambda^2}-\sum_{i=1}^{n}\frac{2(\alpha+1)(1-e^{-\beta x_i^\gamma})}{\lambda^3(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))}-\frac{2n\alpha(1+\lambda^{-1})^{-(\alpha+1)}}{\lambda^3(1-(1+\lambda^{-1})^{-\alpha})}+\sum_{i=1}^{n}\frac{(\alpha+1)(1-e^{-\beta x_i^\gamma})^2}{\lambda^4(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2}+\frac{\alpha(\alpha+1)(1+\lambda^{-1})^{-(\alpha+2)}}{\lambda^4(1-(1+\lambda^{-1})^{-\alpha})}+\frac{n\alpha^2(1+\lambda^{-1})^{-2(\alpha+1)}}{\lambda^4(1-(1+\lambda^{-1})^{-\alpha})^2},$$

$$I_{\lambda,\beta}=\frac{\partial^2 lnL}{\partial\lambda\partial\beta} = -\sum_{i=1}^{n}\frac{(\alpha+1)x_i^\gamma e^{-\beta x_i^\gamma}(1-e^{-\beta x_i^\gamma})}{\lambda^3(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2}+\sum_{i=1}^{n}\frac{(\alpha+1)x_i^\gamma e^{-\beta x_i^\gamma}}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))},$$

$$I_{\lambda,\gamma}=\frac{\partial^2 lnL}{\partial\lambda\partial\gamma} = \sum_{i=1}^{n}\frac{(\alpha+1)\beta lnx_i x_i^\gamma e^{-\beta x_i^\gamma}}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))}-\sum_{i=1}^{n}\frac{(\alpha+1)\beta lnx_i x_i^\gamma e^{-\beta x_i^\gamma}(1-e^{-\beta x_i^\gamma})}{\lambda^3(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2},$$

$$I_{\beta,\beta}=\frac{\partial^2 lnL}{\partial\beta^2} = -\frac{n}{\beta^2}+\sum_{i=1}^{n}\frac{(\alpha+1)x_i^{2\gamma}e^{-\beta x_i^\gamma}}{\lambda(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))}+\sum_{i=1}^{n}\frac{(\alpha+1)x_i^{2\gamma}e^{-2\beta x_i^\gamma}}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2},$$

$$I_{\beta,\gamma}=\frac{\partial^2 lnL}{\partial\beta\partial\gamma} = -\sum_{i=1}^{n}x_i^\gamma lnx_i+\sum_{i=1}^{n}\frac{(\alpha+1)\beta lnx_i x_i^{2\gamma}e^{-\beta x_i^\gamma}}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2}-\sum_{i=1}^{n}\frac{(\alpha+1)lnx_i x_i^\gamma e^{-\beta x_i^\gamma}[1-\beta x_i^\gamma]}{\lambda(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))},$$

$$I_{\gamma,\gamma}=\frac{\partial^2 lnL}{\partial\gamma^2} = -\frac{n}{\gamma^2}-\sum_{i=1}^{n}\beta x_i^\gamma(lnx_i)^2+\sum_{i=1}^{n}\frac{(\alpha+1)\beta^2(lnx_i)^2x_i^{2\gamma}e^{-2\beta x_i^\gamma}}{\lambda^2(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))^2}-\sum_{i=1}^{n}\frac{(\alpha+1)\beta(lnx_i)^2x_i^\gamma e^{-\beta x_i^\gamma}[1-\beta x_i^\gamma]}{\lambda(1+\lambda^{-1}(1-e^{-\beta x_i^\gamma}))}.$$

and the observed information matrix is given by

$$J(\theta)=-\begin{bmatrix} I_{\alpha,\alpha} & I_{\alpha,\lambda} & I_{\alpha,\beta} & I_{a,\gamma} \\ I_{\lambda,\alpha} & I_{\lambda,\lambda} & I_{\lambda,\beta} & I_{\lambda,\gamma} \\ I_{\beta,\alpha} & I_{\beta,\lambda} & I_{\beta,\beta} & I_{\beta,\gamma} \\ I_{\gamma,\alpha} & I_{\gamma,\lambda} & I_{\gamma,\beta} & I_{\gamma,\gamma} \end{bmatrix} \tag{11}$$

To obtain the asymptotic variance-covariance matrix of $(\hat{\alpha},\hat{\lambda},\hat{\beta},\hat{\gamma})$, one can invert the information matrix and replace the unknown parameters by their mles. So $100(1-\nu)\%$ approximate confidence intervals(CIs) for the parameters

$\alpha, \lambda,\ \beta$ andγ are respectively:
$(\hat{\alpha} - z_{\frac{\nu}{2}}\sqrt{var(\hat{\alpha})}, \hat{\alpha} + z_{\frac{\nu}{2}}\sqrt{var(\hat{\alpha})})$,
$(\hat{\lambda} - z_{\frac{\nu}{2}}\sqrt{var(\hat{\lambda})}, \hat{\lambda} + z_{\frac{\nu}{2}}\sqrt{var(\hat{\lambda})})$,
$(\hat{\beta} - z_{\frac{\nu}{2}}\sqrt{var(\hat{\beta})}, \hat{\beta} + z_{\frac{\nu}{2}}\sqrt{var(\hat{\beta})})$,and
$(\hat{\gamma} - z_{\frac{\nu}{2}}\sqrt{var(\hat{\gamma})}, \hat{\gamma} + z_{\frac{\nu}{2}}\sqrt{var(\hat{\gamma})})$.

3.2 Bayesian Estimation

Let non-informative prior distribution take the form: $\pi_0(\alpha, \lambda, \beta, \gamma) \propto \frac{1}{\alpha\lambda\beta\gamma}$ where α, λ, β and $\gamma > 0$.To compute approximate Bayes estimates, one can generate samples from posterior distributions using the Gibbs sampling procedure. The joint posterior distribution is
$\pi(\alpha, \lambda, \beta, \gamma|x) \propto \pi_0(\alpha, \lambda, \beta, \gamma) \exp l(x, \alpha, \lambda, \beta, \gamma)$,
where $l(x, \alpha, \lambda, \beta, \gamma)$ is the logarithm of the likelihood function is given in (9).When we set $\rho_1 = \log(\beta), \rho_2 = \log(\gamma), \rho_3 = \log(\alpha)$ and $\rho_4 = \log(\lambda)$, the joint prior distribution will be $\pi(\rho_1, \rho_2, \rho_3, \rho_4) \propto$ constant,$-\infty < \rho_1, \rho_2, \rho_3$,and $\rho_4 < \infty$ and the joint posterior distribution take the form

$$\begin{aligned}\pi(\rho_1, \rho_2, \rho_3, \rho_4|x) \quad \propto \quad & \pi(\rho_1, \rho_2, \rho_3, \rho_4)\ n \log[1 - (1 + exp(-\rho_4))^{-exp(\rho 3)}] + \rho_1 \\ & + \rho_2 - \rho_4 + \sum_{i=1}^{n}(exp(\rho_2) - 1)\log[x_i] - \sum_{i=1}^{n} exp(\rho_1) x_i^{exp(\rho_2)} \\ & -(exp(\rho_3) + 1)\log[1 + exp(-\rho_4)(1 - e^{-exp(\rho_1 x_i^{exp(\rho_2)}})],\end{aligned}$$

Posterior summaries such as: the mean; the standard deviation; credible interval and others can be calculated using the WinBUGS software.

3.3 A Numerical Example

Now, we generate a sample of size $n = 100$ from Lomax-W distribution to estimate the four unknown parameters α, λ, β,and γ. For our sample, let $\alpha_0 = 1.5, \lambda_0 = 0.5, \beta_0 = 1.5$,and $\gamma_0 = 2.5$. The MLEs and 95 % confidence intervals for the four parameters α,λ, β and γ are listed in table(1):

Table 1: the MLEs and CIs for the four parameters α,λ, β and γ of the Lomax-W model

Parameter	Estimate	Standard Deviation	95% Confidence Interval
α	4.3200	1.2649	(1.84077, 6.79923)
λ	0.0140	0.01414	(0.11228, 0.16772)
β	2.4200	1.3693	(0.0000, 5.10384)
γ	5.1100	0.4641	(4.20034, 6.01966)

With the same sample and with non-informative uniform priors U(0.00, 3.00), U(0.001, 0.01), U(0.001, 0.10)and U(-4.00, 3.00) for$\rho_1, \rho_2, \rho_3 and \rho_4$ respectively, we generate a set of 10000 Gibbs samples generated after burn in sample of size 1000. As table (2) shows the MC error is less than 5 % of the sample standard deviation which indicating convergence of the algorithm.

Table 2: Summary results for the posterior parameters in the case of the Lomax-W model

Parameter	Estimate	Standard Deviation	MC error	95 % Credible Interval
α	1.0180	0.01637	2.536E-4	(1.0010, 1.0620)
λ	0.6024	0.07521	6.588E-5	(0.4675, 0.7564)
β	1.0070	0.28680	0.002674	(2.8630, 3.9860)
γ	3.4040	0.00251	3.332E-5	(1.0010, 1.0100)

From Tables 1-2, one can note that in most cases the CIs based on maximum likelihood inference is larger than the credible intervals based on the posterior summaries.

4 Conclusion

The Weibull distribution may not give adequate fits in many data. For this reason, among many others, many authors suggested new families of distributions in the hope of adding flexibility in the modeling process. Many of these families are modification, extension or combinations of existing one. In this paper we concerned with one of these families which introduced by Gupta *et al.*(2015) and Gupta *et al.*(2016). Using this family we introduced a new generalization of Weibull distribution which called the Lomax-Weibull distribution.Some properties of the new distribution are investigated and maximum likelihood and Bayesian estimation of the four unknown parameters are discussed.

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