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Decompositions of Balanced Complete Bipartite Graphs into Suns and Stars

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Abstract

Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of a graph G. An L-decomposition of G is an edge-disjoint decomposition of G into positive integer α_i copies of H_i, where $i \in \{1, 2, \ldots, r\}$. Let $S(C_{k/2})$ and S_k denote a sun and a star with k edges, respectively. In this paper, we prove that a balanced complete bipartite graph with $2n$ vertices has a $\{S(C_{k/2}), S_k\}$-decomposition if and only if $8 \leq k \leq n$, $k \equiv 0 \pmod 4$ and $n^2 \equiv 0 \pmod k$.

Mathematics Subject Classification: 05C51

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1 Introduction

Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of a graph G. An L-*decomposition* of G is an edge-disjoint decomposition of G into positive integer α_i copies of H_i, where $i \in \{1, 2, \ldots, r\}$. Furthermore, if each H_i is isomorphic to a graph H, then we say that G has an H-*decomposition*.

For positive integers m and n, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A complete bipartite graph is *balanced* if $m = n$. A k-*star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$. The vertex of degree k in S_k is the *center* of S_k. A k-*cycle*, denoted by C_k, is a cycle of length k. A k-sun $S(C_k)$ is obtained from C_k by adding a pendant edge to each vertex of C_k.

Decompositions of graphs into k-stars have been a important topic of research in graph theory; see [16, 17, 18]. The concept of a sun graph was defined by Harary [9]. Anitha and Lekshmi [4, 5] have decomposed K_{2k} into k-sun, Hamilton cycles, and perfect matchings. Liang and Guo [10, 11] gave the existence spectrum of a k-sun system of order v as $k = 3, 4, 5, 6, 8$ by using a recursive construction. Recently, Fu et al. [6] investigate the problem of the decomposition of complete tripartite graphs into 3-suns and find the necessary and sufficient condition for the existence of a k-sun system of order v in [7, 8].

The study of $\{G, H\}$-decomposition was introduced by Abueida and Daven in [1]. Abueida and Daven [2] investigated the problem of $\{K_k, S_k\}$-decomposition of the complete graph K_n. Abueida and O'Neil [3] settled the existence problem for $\{C_k, S_{k-1}\}$-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Recently, Lee [12, 13] established necessary and sufficient conditions for the existence of a $\{C_k, S_k\}$-decomposition of a complete bipartite graph and $\{P_k, S_k\}$-decomposition of a balanced complete bipartite graph. In this paper, we consider the existence of $\{S(C_{k/2}), S_k\}$-decompositions of the balanced complete bipartite graph, giving necessary and sufficient conditions.

2 Preliminaries

Let G be a graph. The *degree* of a vertex x of G, denoted by $\deg_G x$, is the number of edges incident with x. For $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ and $G - B$ to denote the subgraph of G induced by A and the subgraph of G obtained by deleting B, respectively. When $G_1, G_2 \ldots, G_m$ are graphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_m$ or $\bigcup_{i=1}^{m} G_i$ for the graph with vertex set $\bigcup_{i=1}^{m} V(G_i)$ and edge set $\bigcup_{i=1}^{m} E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^{m} G_i$ expresses the decomposition of G into $G_1, G_2, \ldots, G_m$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. For any vertex x of a digraph G, the *outdegree* $\deg_G^+ x$ (respectively, *indegree* $\deg_G^- x$) of x is the number of arcs incident from (respectively, to) x. The following propositions will be helpful to the proof of our main result.

Proposition 2.1. (Sotteau [15]) *For positive integers m, n and k, the graph $K_{m,n}$ has a C_k-decomposition if and only if m, n and k are even, $k \geq 4$, $\min\{m, n\} \geq k/2$, and $mn \equiv 0 \pmod{k}$.*

Proposition 2.2. (Ma et al. [14]) *For positive integers n and k, the graph obtained by deleting a 1-factor from $K_{n,n}$ has a C_k-decomposition if and only if n is odd, k is even, $4 \leq k \leq 2n$, and $n(n-1) \equiv 0 \pmod{k}$.*

Proposition 2.3. (Yamamoto et al. [18]) *For integers m and n with $m \geq n \geq 1$, the graph $K_{m,n}$ has an S_k-decomposition if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k \\ mn \equiv 0 \pmod{k} & \text{if } n \geq k. \end{cases}$$

For positive integers k and n with $1 \leq k \leq n$, the *crown* $C_{n,k}$ is the graph with vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i b_j : i = 1, 2, \ldots, n, j \equiv i+1, i+2, \ldots, i+k \pmod{n}\}$. For the edge $a_i b_j$ in $C_{n,k}$, the *label* of $a_i b_j$ is $j - i \pmod{n}$.

A *matching* in a graph G is a subset of edges of G that share no vertices. A 1-*factor* M in a graph G is a matching such that every vertex of G is incident with one of the edges of M. Trivially, the edges labeled i $(1 \leq i \leq k)$ form a 1-factor in $C_{n,k}$.

G is a bipartite graph with bipartition $(\{(a_0, a_1, \ldots, a_{m-1}\}, \{(b_0, b_1, \ldots, b_{n-1}\})$. Then $G^{r\times s}$ is a bipartite graph with bipartition $(\{(a_{\alpha,0} a_{\alpha,1}, \ldots, a_{\alpha,m-1}\}, \{(b_{\beta,0}, b_{\beta,1}, \ldots, b_{\beta,n-1}\})$ and $E(G^{r\times s}) = \{a_{\alpha,i} b_{\beta,j} : a_i b_j \in E(G)\}$ for $0 \leq \alpha \leq r-1$, $0 \leq \beta \leq s-1$.

Throughout this paper, the subscripts of a_i and b_j will always be taken modulo m and n, respectively.

Lemma 2.4. *$(C_{n,3})^{2\times 4}$ has an $S(C_{2n})$-decomposition.*

Proof. Let $(C_{n,3})^{2\times 4} = \bigcup_{\alpha=0}^{1} \bigcup_{\beta=0}^{3} G_{\alpha,\beta}$, where $G_{\alpha,\beta}$ is a crown $C_{n,3}$ with bipartition $(\{a_{\alpha,0}, a_{\alpha,1}, \ldots, a_{\alpha,n-1}\}, \{b_{\beta,0}, b_{\beta,1}, \ldots, b_{\beta,n-1}\})$ for $\alpha \in \{0,1\}$, $\beta \in \{0,1,2,3\}$. Let $C_{\alpha,\beta}$ be the union of edges labeled 1 and edges labeled 2 of $G_{\alpha,\beta}$, $F_{\alpha,\beta}$ be the union of edges labeled 3 of $G_{\alpha',\beta}$ and edges labeled $\beta+1$ of $G_{\alpha,3}$, where $\alpha \in \{0,1\}$, $\beta \in \{0,1,2\}$ and $\alpha' \neq \alpha$. Since $C_{\alpha,\beta}$ is a $2n$-cycle of $G_{\alpha,\beta}$, $F_{\alpha,\beta}$ are two copies of 1-factors of $G_{\alpha',\beta}$ and $G_{\alpha,3}$, it follows that $C_{\alpha,\beta} \cup F_{\alpha,\beta}$ is a $2n$-sun. Hence $(C_{n,3})^{2\times 4}$ can be decomposed into 6 copies of $S(C_{2n})$. □

Lemma 2.5. *$(C_{n,2\ell})^{2\times 2}$ has an $S(C_{2n})$-decomposition.*

Proof. Let $(C_{n,2\ell})^{2\times 2} = \bigcup_{\alpha=0}^{1} \bigcup_{\beta=0}^{1} G_{\alpha,\beta}$, where $G_{\alpha,\beta}$ is a crown $C_{n,2\ell}$ with bipartition $(\{a_{\alpha,0}, a_{\alpha,1}, \ldots, a_{\alpha,n-1}\}, \{b_{\beta,0}, b_{\beta,1}, \ldots, b_{\beta,n-1}\})$ for $\alpha, \beta \in \{0,1\}$. For $i = 0, 1, \ldots, \ell-1$, let

$C_{0,0}^{(i)}$ be the union of edges labeled $2i+1$ and edges labeled $2i+2$ of $G_{0,0}$;

$C_{1,1}^{(i)}$ be the union of edges labeled $2i+1$ and edges labeled $2i+2$ of $G_{1,1}$;

$F_{0,0}^{(i)}$ be the union of edges labeled $2i+1$ of $G_{1,0}$ and edges labeled $2i+2$ of $G_{0,1}$;

$F_{1,1}^{(i)}$ be the union of edges labeled $2i+2$ of $G_{1,0}$ and edges labeled $2i+1$ of $G_{0,1}$.

Note that $C_{0,0}^{(i)}$ and $C_{1,1}^{(i)}$ are $2n$-cycle of $G_{0,0}$ and $G_{1,1}$, respectively. In additional, $F_{0,0}^{(i)}$ and $F_{1,1}^{(i)}$ are two copies of 1-factors of $G_{0,1}$ and $G_{1,0}$. It follows that $C_{0,0}^{(i)} \cup F_{0,0}^{(i)}$ is a $2n$-sun, so is $C_{1,1}^{(i)} \cup F_{1,1}^{(i)}$ for $i = 0, 1, \ldots, \ell-1$. Hence $(C_{n,2\ell})^{2\times 2}$ can be decomposed into 2ℓ copies of $S(C_{2n})$. □

Lemma 2.6. *If $k \equiv 0 \pmod 4$ with $k \geq 8$, then $K_{k/2,k}$ can be decomposed into $k/2$ copies of $S(C_{k/2})$.*

Proof. Note that

$$K_{k/2,k} = \begin{cases} 2(C_{k/4,k/4})^{2\times 2}, & \text{if } k/4 \text{ is even,} \\ 2(C_{k/4,k/4-3})^{2\times 2} \cup (C_{k/4,3})^{2\times 4}, & \text{if } k/4 \text{ is odd.} \end{cases}$$

By Lemmas 2.4 and 2.5, $K_{k/2,k}$ can be decomposed into $k/2$ copies of $S(C_{k/2})$. □

3 Main results

We first give necessary conditions for a $\{S(C_{k/2}), S_k\}$-decomposition of $K_{n,n}$.

Lemma 3.1. *If $K_{n,n}$ has a $\{S(C_{k/2}), S_k\}$-decomposition, then $8 \leq k \leq n$, $k \equiv 0 \pmod 4$ and $n^2 \equiv 0 \pmod k$.*

Proof. Since bipartite graphs contain no odd cycle, $k/2$ is even. It follows that $k \equiv 0 \pmod 4$. In addition, the minimum length of a cycle and the maximum size of a star in $K_{n,n}$ are 4 and n, respectively, we have $8 \leq k \leq n$. Finally, the size of each member in the decomposition is k and $|E(K_{n,n})| = n^2$; thus $n^2 \equiv 0 \pmod k$. □

We now show that the necessary conditions are also sufficient. The proof is divided into cases $n = k$, $k < n < 2k$, and $n \geq 2k$, which are treated in Lemmas 3.2, 3.3, and 3.4, respectively.

Lemma 3.2. *If $k \equiv 0 \pmod 4$ with $k \geq 8$, then $K_{k,k}$ has a $\{S(C_{k/2}), S_k\}$-decomposition.*

Proof. Since $K_{k,k} = K_{k/2,k} \cup K_{k/2,k}$, by Proposition 2.3 and 2.6, $K_{k/2,k}$ has an S_k-decomposition and $K_{k/2,k}$ has an $S(C_{k/2})$-decomposition, respectively. Hence $K_{k,k}$ has a $\{S(C_{k/2}), S_k\}$-decomposition. □

Lemma 3.3. *Let k be a multiple of 4 and let n be a positive integer with $8 \leq k < n < 2k$. If n^2 is divisible by k, then $K_{n,n}$ has a $\{S(C_{k/2}), S_k\}$-decomposition.*

Proof. Let $n = k+r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $t = r^2/k$. Since $k \mid n^2$, we have $k \mid r^2$, which implies that t is a positive integer. Let $K_{n,n}$ be the balanced complete bipartite graph with bipartition $(A_0 \cup A_1 \cup A_2, B_0 \cup B_1 \cup B_2)$, where $A_i = \{a^{(i)}_{0,0}, a^{(i)}_{0,1}, \ldots, a^{(i)}_{0,k/4-1}, a^{(i)}_{1,0}, a^{(i)}_{1,1}, \ldots, a^{(i)}_{1,k/4-1}\}$, $B_j = \{b^{(j)}_{0,0}, b^{(j)}_{0,1}, \ldots, b^{(j)}_{0,k/4-1}, b^{(j)}_{1,0}, b^{(j)}_{1,1}, \ldots, b^{(j)}_{1,k/4-1}\}$ for $i, j \in \{0,1\}$, $A_2 = \{a_k, a_{k+1}, \ldots, a_{k+r-1}\}$ and $B_2 = \{b_k, b_{k+1}, \ldots, b_{k+r-1}\}$.

Let $G_i = K_{n,n}[A_i \cup \{B_0 \cup B_1\}]$ for $i = 0, 1$, $F = K_{n,n}[A_2 \cup (B_0 \cup B_1)]$, and $H = K_{n,n}[A \cup B_2]$. Clearly $K_{n,n} = G_0 \cup G_1 \cup F \cup H$. Note that $G_i = ((K_{k/4,k/4})^{2\times 2})^{1\times 2}$ for $i = 0, 1$, H is isomorphic to $K_{n,r}$, and F is isomorphic to $K_{r,k}$, which has an S_k-decomposition by Proposition 2.3. Let $p_0 = \lceil t/2\rceil$, $p_1 = \lfloor t/2\rfloor$ and $\alpha_i = 3\lceil p_i/2\rceil + \lfloor p_i/2\rfloor$, $\beta_i = 3\lfloor p_i/2\rfloor + \lceil p_i/2\rceil$ for $i \in \{0, 1\}$. In the following, we will show that G_0 can be decomposed into p_0 copies of $S(C_{k/2})$, $k/4$ copies of $S_{k-\alpha_0}$ and $k/4$ copies of $S_{k-\beta_0}$, G_1 can be decomposed into p_1 copies of $S(C_{k/2})$, $k/4$ copies of $S_{k-\alpha_1}$ and $k/4$ copies of $S_{k-\beta_1}$, H can be decomposed into $k/4$ copies of S_{α_0}, $k/4$ copies of S_{β_0}, $k/4$ copies of S_{α_1}, $k/4$ copies of S_{β_1} and r copies of S_k.

We first show the required decomposition of G_0 and G_1. Since $r < k$, we have $r \geq t + 1$. Furthermore, since $(t, t + 1) = 1$ that $r \geq t + 2$. Thus, $p_0 = \lceil t/2\rceil \leq (t + 1)/2 < r/2 < k/2$. This assures us that there exist p_0 edge-disjoint k-sun graphs in G_0 and p_1 edge-disjoint k-sun graphs in G_1 by Lemma 2.6, respectively. Suppose that $Q_{0,0}, Q_{0,1}, \ldots, Q_{0,p_0-1}$ and $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,p_1-1}$ are edge-disjoint k-sun graphs in G_0 and G_1, respectively. Let $W_i = G_i - E(\bigcup_{h=0}^{p_i-1} Q_{i,h})$ and $X^{(i)}_{s,t} = W_i[\{a^{(i)}_{s,t}\} \cup (B_0 \cup B_1)]$ where $i, s \in \{0, 1\}$ and $t \in \{0, 1, \ldots, k/4-1\}$. Since $\deg_{G_i} a^{(i)}_{s,t} = k$ and each $Q_{i,h}$ uses three edges incident with $a^{(i)}_{s,t}$ and one edge incident with $a^{(j)}_{s,t}$ for $s, i, j \in \{0, 1\}$, $i \neq j$, $t \in \{0, 1, \ldots, k/4 - 1\}$, we have

$$\deg_{W_i} a^{(i)}_{s,t} = \begin{cases} k - \alpha_i, & \text{if } s = 0, \\ k - \beta_i, & \text{if } s = 1. \end{cases}$$

Then

$$X^{(i)}_{s,t} = \begin{cases} S_{k-\alpha_i}, & \text{if } s = 0, \\ S_{k-\beta_i}, & \text{if } s = 1 \end{cases}$$

with the center at $A_0 \cup A_1$.

Next we show the required star-decompositions of H. Equivalently we need show that there exists an orientation of H such that, $i \in \{0, 1\}$, $t \in \{0, 1, \ldots, k/4 - 1\}$, and $w \in \{k, k + 1, \ldots, k + r - 1\}$,

$$\deg^+_H a^{(i)}_{s,t} = \begin{cases} \alpha_i, & \text{if } s = 0, \\ \beta_i, & \text{if } s = 1, \end{cases} \tag{1}$$

$$\deg^+_H b_w = k. \tag{2}$$

We begin the orientation. For $t = 0, 1, \ldots, k/4 - 1$, the edges

$a^{(0)}_{0,t} b_{k+\alpha_0 t}, a^{(0)}_{0,t} b_{k+\alpha_0 t+1}, \ldots, a^{(0)}_{0,t} b_{k+\alpha_0 t+\alpha_0-1}$,

$a^{(0)}_{1,t} b_{k(1+\frac{\alpha_0}{4})+\beta_0 t}, a^{(0)}_{1,t} b_{k(1+\frac{\alpha_0}{4})+\beta_0 t+1}, \ldots, a^{(0)}_{1,t} b_{k(1+\frac{\alpha_0}{4})+\beta_0 t+\beta_0-1}$,

$a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4})+\alpha_1 t}, a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4})+\alpha_1 t+1}, \ldots, a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4})+\alpha_1 t+\alpha_1-1}$,

$a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4}+\frac{\alpha_1}{4})+\beta_1 t}, a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4}+\frac{\alpha_1}{4})+\beta_1 t+1}, \ldots, a^{(1)}_{0,t} b_{k(1+\frac{\alpha_0}{4}+\frac{\beta_0}{4}+\frac{\alpha_1}{4})+\beta_1 t+\beta_1-1}$

are oriented from $A_0 \cup A_1$, where the subscripts of b's are taken modulo r in the set of numbers $\{k, k+1, \ldots, k+r-1\}$. Since $\max\{\alpha_0, \beta_0, \alpha_1, \beta_1\} = \alpha_0 \leq 2p_0 + 1 \leq t + 2 \leq r$, this assures us that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from $\{b_k, b_{k+1}, \ldots, b_{k+r-1}\}$ to $A_0 \cup A_1$.

From the construction of the orientation, it is easy to see that (1) is satisfied, and for all $w, w' \in \{k, k+1, \ldots, k+r-1\}$, we have

$$|\deg_H^- b_w - \deg_H^- b_{w'}| \leq 1. \tag{3}$$

Thus we only need to check (2).

Since $\deg_H^+ b_w + \deg_H^- b_w = k + r$ for $w \in \{k, k+1, \ldots, k+r-1\}$, it follows from (3) that $|\deg_H^+ b_w - \deg_H^+ b_{w'}| \leq 1$ for $w, w' \in \{k, k+1, \ldots, k+r-1\}$. Furthermore,

$$\begin{aligned}
\sum_{w=k}^{k+r-1} \deg_H^+ b_w &= |E(K_{n,r})| - \sum_{i=0}^{1}\sum_{s=0}^{1}\sum_{t=0}^{k/4-1} \deg_H^+ a_{s,t}^{(i)} \\
&= (k+r)r - (\alpha_0 + \beta_0 + \alpha_1 + \beta_1)(k/4) \\
&= (k+r)r - (4p_0 + 4p_1)(k/4) \\
&= (k+r)r - tk \\
&= (k+r)r - r^2 \\
&= kr
\end{aligned}$$

Thus $\deg_H^+ b_w = k$ for $w \in \{k, k+1, \ldots, k+r-1\}$. This proves (2). Hence there exists a decomposition $\mathscr{D}$ of H into $k/4$ copies of S_{α_0}, $k/4$ copies of S_{β_0}, $k/4$ copies of S_{α_1} and $k/4$ copies of S_{β_1} with centers in $\{a_{0,0}^{(0)}, a_{0,1}^{(0)}, \ldots, a_{0,k/4-1}^{(0)}\}$, $\{a_{1,0}^{(0)}, a_{1,1}^{(0)}, \ldots, a_{1,k/4-1}^{(0)}\}$, $\{b_{0,0}^{(1)}, b_{0,1}^{(1)}, \ldots, b_{0,k/4-1}^{(1)}\}$ and $\{b_{1,0}^{(1)}, b_{1,1}^{(1)}, \ldots, b_{1,k/4-1}^{(1)}\}$, respectively, as well as r copies of S_k with centers in $\{b_k, b_{k+1}, \ldots, b_{k+r-1}\}$. Let $X_{0,t}^{'(i)}$ be the α_i-star with center $a_{0,t}^{(i)}$ and $X_{1,t}^{'(i)}$ be the β_i-star with centers $a_{1,t}^{(i)}$ in $\mathscr{D}$. Note that $X_{s,t}^{(i)} \cup X_{s,t}^{'(i)}$ is a k-star for $i, s \in \{0, 1\}$ and $t \in \{0, 1, \ldots, k/4-1\}$. Thus $K_{n,n}$ can be decomposed into $p_0 + p_1 = t$ copies of $S(C_{k/2})$ and $(k+r) + r = k + 2r$ copies of S_k. This completes the proof. □

Lemma 3.4. *Let k be a multiple of 4 and let n be a positive integer with $8 \leq k \leq n/2$. If n^2 is divisible by k, then $K_{n,n}$ has a $\{S(C_{k/2}), S_k\}$-decomposition.*

Proof. Let $n = qk + r$ where q and r are integers with $0 \leq r < k$. From the assumption of $k \leq n/2$, we have $q \geq 2$. Note that

$$K_{n,n} = K_{qk+r,qk+r} = K_{(q-1)k,(q-1)k} \cup K_{k+r,(q-1)k} \cup K_{(q-1)k,k+r} \cup K_{k+r,k+r}.$$

Trivially, $|E(K_{(q-1)k,(q-1)k})|$, $|E(K_{k+r,(q-1)k})|$ and $|E(K_{(q-1)k,k+r})|$ are multiples of k. Thus $(k+r)^2 \equiv 0 \pmod{k}$ from the assumption that n^2 is divisible by

k. The case of $r = 0$, by Lemma 3.2, we obtain that $K_{k,k}$ has a $\{S(C_{k/2}), S_k\}$-decomposition. The other case of $r \neq 0$, by Lemma 3.3, $K_{k+r,k+r}$ has a $\{S(C_{k/2}), S_k\}$-decomposition for $0 < r < k$. On the other hand, by Proposition 2.3, $K_{(q-1)k,(q-1)k}$, $K_{k+r,(q-1)k}$ and $K_{(q-1)k,k+r}$ have S_k-decomposition. Hence there exists a $\{S(C_{k/2}), S_k\}$-decomposition of $K_{n,n}$. □

Now we are ready for the main result. It is obtained by combining Lemmas 3.1, 3.2, 3.3 and 3.4.

Theorem 3.5. *$K_{n,n}$ has a $\{S(C_{k/2}), S_k\}$-decomposition if and only if $8 \leq k \leq n$, $k \equiv 0 \pmod 4$ and $n^2 \equiv 0 \pmod k$.*

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