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Nevanlinna-Pick Problem on Reproducing Kernel Spaces

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Abstract

The Nevanlinna-Pick problem on a reproducing kernel space is known to be solvable if and only if the Nevanlinna-Pick matrix is nonnegative. In this paper, we give a constructive proof showing the determinant of the matrix is nonnegative is already equivalent to the solvability of the problem. Then we obtain formulas for the minimal norm among solutions and consequences concerning interpolating values.

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1 Introduction

Let D be the open unit disk in the complex plane C and $H^\infty(D)$ be the Banach space of bounded holomorphic functions on D. In [5] G. Pick and in [4] R. Nevanlinna independently studied the problem of finding a condition that determines whether given finitely many distinct nodes $z_1, z_2, \ldots, z_n \in D$ and values $w_1, w_2, \ldots, w_n \in C$, there exists an f in the closed unit ball of $H^\infty(D)$ such that $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$. Pick proved that this interpolation problem is solvable if and only if the matrix M is nonnegative, where for $i, j = 1, 2, \ldots, n$,

$$M_{i,j} = \frac{1 - \overline{w_i} w_j}{1 - \overline{z_i} z_j}$$

(i.e. the quadratic form associated with M is positive definite or positive semi-definite). There are many different proofs of this result based on different approaches (see for example [3], [6], [7]). When the interpolation problem is solvable, the following consequences are known (see [2]):

(a) there is a unique solution if and only if $\det M = 0$,
(b) for every $z_{n+1} \in D \setminus \{z_1, \ldots, z_n\}$, the set of possible values at z_{n+1}, namely

$$W = \{f(z_{n+1}) : f \in H^\infty(D), \|f\| \leq 1, f(z_i) = w_i, i = 1, \ldots, n\},$$

is a closed disk in D,
(c) for every w_{n+1} in the boundary of the disk W, there is a unique function $f \in H^\infty(D)$ such that $f(z_i) = w_i$ for $i = 1, \ldots, n+1$ and
(d) the minimal norm interpolation function is always of the form a constant times a Blaschke product.

In this paper, we will study the above interpolation problem in *reproducing kernel spaces* on D. These are Hilbert spaces of holomorphic functions on D, where evaluation at any point of D induces a bounded linear functional. So for $x \in D$, there exists a unique $K_x \in H$ such that $f(x) = \langle f, K_x \rangle$ for all $f \in H$. This K_x is called the *reproducing kernel* at x. We will also assume $\{K_x : x \in D\}$ is linearly independent, which is satisfied by most common function spaces like the Hardy space $H^2(D)$. The following theorems are our main results.

Theorem 1 *Let H be a reproducing kernel space on D. Let $z_1, z_2, \ldots, z_n \in D$ be distinct and $w_1, w_2, \ldots, w_n \in C$. Suppose the reproducing kernels $K_1, K_2, \ldots, K_n$ at $z_1, z_2, \ldots, z_n$, respectively, are linearly independent. Then the following are equivalent:*

(1) There exists an $f \in H$ such that $\|f\| \leq 1$ and $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$;
(2) the matrix $M = (\langle K_i, K_j \rangle - \overline{w_i} w_j)_{i,j=1,2,\ldots,n}$ is nonnegative;
(3) $\det M \geq 0$, where M is the matrix in (2)

Theorem 2 *If the interpolation problem is solvable, then the minimal norm of all such interpolation functions is given by the expression*

$$\sqrt{1 - (\det M / \det G)},$$

where G is the Gram matrix $(\langle K_i, K_j \rangle)$, and the interpolation function with minimal norm is a linear combination of the reproducing kernels at the nodes.

In the first theorem, the equivalence of conditions (1) and (2) can be obtained by modifying a theorem of E. Helly and F. Riesz as done in [1], where the problem is solved theoretically without further discussions of analogous consequences like (a) to (d) above. Note condition (2) implies that the possible values at an addition node fill a closed *convex* set only. The interesting part of the first theorem above is that the apparently weaker condition (3) actually implies condition (1) and it is condition (3) that allows us to deduce that the values at an addition node fill a closed disk.

In the second theorem, we provide a description of the minimal interpolation situation with a formula for the minimal norm in terms of the interpolation data z_i's and w_i's. It is interesting that no such formula seems to be known for the original Nevanlinna-Pick problem. Using the formula for the minimal norm, we can easily deduce the analogous consequences. Below we will give constructive proofs of the theorems above, without using the Helly-Riesz theorem for proving the first theorem in particular.

2 Proofs of the Theorems

To prove the theorems, we will first establish a formula for the determinant of a rank one perturbation of a Hermitian matrix. Below we will use the notation cof M_{ij} to denote the cofactor of the (i, j)-entry of the matrix M.

Proposition 1 *Let L be an $n \times n$ positive Hermitian matrix and V be the $1 \times n$ matrix $(w_1 \quad w_2 \quad \cdots \quad w_n)$, then*

$$\det(L - V^*V) = \det L - \sum_{i,j=1}^{n} (\text{cof } L_{ij})\overline{w_i}w_j.$$

Proof. Since L is positive Hermitian, there exist positive $\lambda_1, \dots, \lambda_n$ and a unitary matrix U such that

$$U^*LU = \text{diag}(\lambda_1, \dots, \lambda_n).$$

Let $VU = (z_1 \quad z_2 \quad \cdots \quad z_n)$, where

$$z_k = \sum_{i=1}^{n} w_i U_{ik} \quad \text{for } k = 1, 2, \dots, n.$$

Let $X = (z_1/\sqrt{\lambda_1} \quad z_2/\sqrt{\lambda_2} \quad \cdots \quad z_n/\sqrt{\lambda_n})$ and $\widehat{\lambda_k}$ be the product of $\lambda_1, \lambda_2, \dots, \lambda_n$ omitting λ_k. Then

$$\begin{aligned} \det(L - V^*V) &= \det(\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) - (VU)^*(VU)) \\ &= \lambda_1 \cdots \lambda_n \det(I - X^*X) \\ &= \lambda_1 \cdots \lambda_n \det(I - XX^*) \end{aligned}$$

$$
\begin{aligned}
&= \lambda_1 \cdots \lambda_n \Big(1 - \sum_{k=1}^{n} \frac{|z_k|^2}{\lambda_k}\Big) \\
&= \lambda_1 \lambda_2 \cdots \lambda_n - \sum_{k=1}^{n} \widehat{\lambda_k} |z_k|^2 \\
&= \det L - \sum_{k=1}^{n} \widehat{\lambda_k} \left| \sum_{i=1}^{n} w_i U_{ik} \right|^2 \\
&= \det L - \sum_{i,j=1}^{n} \alpha_{ij} \overline{w_i} w_j,
\end{aligned}
$$

where $\alpha_{ij} = \sum_{k=1}^{n} \widehat{\lambda_k} \overline{U_{ik}} U_{jk}$ does not depend on the w_i's. So to find α_{ii}, we may set $w_i = 1$, $w_k = 0$ for all $k \neq i$ and we find $\alpha_{ii} = \text{cof } L_{ii}$. For $i \neq j$, we set $w_i = 1, w_j = 1$, $w_k = 0$ for $k \neq i, j$ and we find

$$\alpha_{ji} + \alpha_{ij} = \text{cof } L_{ji} + \text{cof } L_{ij}.$$

Then we set $w_i = \sqrt{-1}, w_j = 1$ and $w_k = 0$ for $k \neq i, j$ and we find

$$\alpha_{ji} - \alpha_{ij} = \text{cof } L_{ji} - \text{cof } L_{ij}.$$

Therefore, $\alpha_{ij} = \text{cof } L_{ij}$.

Now we are ready to the prove the theorems stated above.

Proof of Theorem 1. (1) $\Rightarrow$ (2) If such an f exists, then the quadratic form associated with M is

$$
\begin{aligned}
Q(t_1, t_2, \ldots, t_n) &= \sum_{i,j=1}^{n} (\langle K_i, K_j \rangle - \overline{w_i} w_j) t_i \overline{t_j} \\
&= \sum_{i,j=1}^{n} \langle K_i, K_j \rangle t_i \overline{t_j} - \sum_{i,j=1}^{n} \overline{\langle f, K_i \rangle} \langle f, K_j \rangle t_i \overline{t_j} \\
&= \Big\|\sum_{i=1}^{n} t_i K_i\Big\|^2 - \Big|\langle f, \sum_{i=1}^{n} t_i K_i \rangle\Big|^2 \geq 0
\end{aligned}
$$

for all $t_1, t_2, \ldots, t_n \in C$ by the Cauchy-Schwarz inequality.

(2) $\Rightarrow$ (3) Well-known.

(3) $\Rightarrow$ (1) Define $f = \sum_{i=1}^{n} c_i K_i$ so that

$$w_i = f(z_i) = \langle \sum_{j=1}^{n} c_j K_j, K_i \rangle = \sum_{j=1}^{n} c_j \langle K_j, K_i \rangle \quad \text{for } i = 1, 2, \ldots, n.$$

Since $K_1, K_2, \ldots, K_n$ are linearly independent, the determinant of the Gram matrix $G = (\langle K_i, K_j\rangle)$ is positive. By Cramer's rule, we have

$$c_j = \frac{1}{\det G}\sum_{i=1}^{n} w_i T_{ij} \quad \text{for} \quad j = 1, 2, \ldots, n,$$

where T_{ij} is the cofactor of the (i, j)-entry of the transpose of G. Note $\overline{T_{ij}} =$ cof G_{ij}. Now the inequality $\|f\|^2 = \sum_{l,m=1}^{n} c_l \overline{c_m} G_{lm} \le 1$ is equivalent to

$$\|f\|^2(\det G)^2 = \sum_{l,m=1}^{n}\Big(\sum_{j=1}^{n} w_j T_{jl}\Big)\Big(\sum_{i=1}^{n}\overline{w_i}\overline{T_{im}}\Big)G_{lm} \le (\det G)^2.$$

Interchanging the summation i, j with the summation l, m, simplifying, then using the formula in the proposition, we get

$$\begin{aligned}
\|f\|^2(\det G)^2 &= \sum_{i,j=1}^{n}\overline{w_i}w_j\sum_{l,m=1}^{n}\overline{T_{im}}T_{jl}G_{lm} = \sum_{i,j=1}^{n}\overline{w_i}w_j\Big(\sum_{m=1}^{n}\overline{T_{im}}\sum_{l=1}^{n}T_{jl}G_{lm}\Big)\\
&= \sum_{i,j=1}^{n}\overline{w_i}w_j\sum_{m=1}^{n}\overline{T_{im}}(\delta_{j,m}\det G) = (\det G)\sum_{i,j=1}^{n}\overline{w_i}w_j\overline{T_{ij}}\\
&= (\det G)\sum_{i,j=1}^{n}(\text{cof } G_{ij})\overline{w_i}w_j = (\det G)(\det G - \det M).
\end{aligned}$$

Since $\det G > 0$, we see that $\|f\| \le 1$ is equivalent to $\det M \ge 0$.

Proof of Theorem 2. The function f in the proof of (3) $\Rightarrow$ (1) above has norm $\sqrt{1 - (\det M/\det G)}$. Since

$$\{h \in H : h(z_i) = w_i, i = 1, \ldots, n\} = f + \{g \in H : g(z_i) = 0, i = 1, \ldots, n\}$$

and f is in $\text{span}\{K_1, K_2, \ldots, K_n\} = \{g \in H : g(z_i) = 0, i = 1, \ldots, n\}^{\perp}$, f is the minimal norm solution.

One curious fact the formula provides is that $\det G$ is always greater than or equal to $\det M$. So measuring by determinant, the interpolation problem $f(z_i) = 0$ is *extremal* among all solvable Nevanlinna-Pick interpolation problems.

With the second theorem providing the minimal interpolation situation, we will next deduce the remaining usual consequences of the Nevanlinna-Pick problem.

Corollary 1 *For a reproducing kernel space H, if the Nevanlinna-Pick problem $f(z_i) = w_i$ for $i = 1, \ldots, n$, and $f \in H$ with $\|f\| \le 1$ is solvable, then*

(1) *there exists a unique solution if and only if* $\det M = 0$,

(2) *for every $z_{n+1} \in D \setminus \{z_1, \ldots, z_n\}$, the set $W = \{f(z_{n+1}) : f \in H, \|f\| \le 1, f(z_i) = w_i$ for $i = 1, \ldots, n\}$ is a closed disk and*

(3) *for every w_{n+1} in the boundary of W, there is a unique function $f \in H$ such that $\|f\| \le 1$ and $f(z_i) = w_i$ for $i = 1, \ldots, n+1$.*

Proof. For (1), there exists a unique solution if and only if the minimal norm of all interpolation functions is 1, which is equivalent to $\det M = 0$ by the formula in the second theorem.

For (2), if z_{n+1} is given, then we can apply the formula in the proposition to express $\det M \ge 0$ as

$$\sum_{i,j=1}^{n+1} (\text{cof } G_{ij})\overline{w_i} w_j \le \det G \qquad (*)$$

for the $(n+1) \times (n+1)$ matrix M. Observe that the coefficient of the $|w_{n+1}|^2$ term in the inequality is the determinant of the Gram matrix for $K_1, K_2, \ldots, K_n$, which is positive. When $z_1, z_2, \ldots, z_{n+1}, w_1, w_2, \ldots, w_n$ are fixed, the inequality above can be rearranged into the form $|w_{n+1} - c| \le r$, where

$$c = -\frac{1}{\text{cof } G_{n+1,n+1}} \sum_{j=1}^{n} (\text{cof } G_{n+1,j}) w_j$$

and

$$r = \frac{1}{\text{cof } G_{n+1,n+1}} \sqrt{\det G - \sum_{i,j=1}^{n} (\text{cof } G_{ij} - \text{cof } G_{n+1,n+1} \text{cof } G_{n+1,j} \overline{\text{cof } G_{n+1,i}}) \overline{w_i} w_j}.$$

For (3), when w_{n+1} is in the boundary of W, we have the equality case of (*), which traces back to $\det M = 0$ and hence by (1), the solution is unique.

3 Remarks

From the solvability conditions of the Nevanlinna-Pick problems for $H^\infty(D)$ and $H^2(D)$, we will try to make a conjecture for the Nevanlinna-Pick problem of $H^p(D)$. For the Hardy space $H^2(D)$, if $z_1, z_2, \ldots, z_n \in D$ are distinct, then the reproducing kernels $K_1, K_2, \ldots, K_n$ at $z_1, z_2, \ldots, z_n$ are linearly independent because their poles are distinct. On $H^2(D)$, we have $K_i(z) = 1/(1-\overline{z_i}z)$. So

$$M = \left(\frac{1-(1-\overline{z_i}z_j)\overline{w_i}w_j}{1-\overline{z_i}z_j}\right)_{i,j=1,2,\ldots,n}.$$

Comparing the Nevanlinna-Pick matrices for $H^\infty(D)$ and $H^2(D)$, we arrive at the natural conjecture that

$$M_p = \left(\frac{1-(1-\overline{z_i}z_j)^{2/p}\overline{w_i}w_j}{1-\overline{z_i}z_j}\right)_{i,j=1,2,\ldots,n} \geq 0$$

is equivalent to the solvability of the Nevanlinna-Pick problem on $H^p(D)$ $(0 < p \leq \infty)$. Here the principal branch is taken for the entries of the matrix.

As a small piece of evidence, we point out that the conjecture is true for one node. For $n = 1$, if there is $f \in H^\infty(D)$ with $\|f\|_p \leq 1$ and $f(z_1) = w_1$, then by the well-known inequality $|f(z)| \leq (1-|z|^2)^{-1/p}\|f\|_p$, we get $|w_1| \leq (1-|z_1|^2)^{-1/p}$, which is equivalent to $M_p \geq 0$. Conversely, if $M_p \geq 0$, then the function

$$f(z) = w_1\Big(\frac{1-|z_1|^2}{1-\overline{z_1}z}\Big)^{2/p} \in H^p(D)$$

satisfies

$$f(z_1) = w_1 \quad \text{and} \quad \|f\|_p = |w_1|(1-|z_1|^2)^{1/p} \leq 1.$$

Curiously, $\sqrt{1-(\det M_p/\det G)}$ also equals $|w_1|(1-|z_1|^2)^{1/p}$. However, for $n > 1$, the formula for the least norm will likely be more complicated due to the factor $(1-\overline{z_i}z_j)^{2/p}$ in the matrix M_p.

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