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On k-Fibonacci and k-Lucas Trigintaduonions

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Abstract

The trigintaduonions form a 32-dimensional Cayley-Dickson algebra. In this paper, we introduce the k- Fibonacci and k-Lucas trigintaduonions. Moreover, we give some properties of these trigintaduonions and derive relationships between them.

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1. Introduction

The k-Fibonacci numbers appear in many fields of science (see, e.g., [14]). The k-Fibonacci number $F_{k,n}$ is defined by the recurrence relation

$$F_{k,0} = 0, F_{k,1} = 1, F_{k,n+1} = kF_{k,n} + F_{k,n-1}; n \geq 1.$$

Another important sequence is the k-Lucas sequence. This sequence is defined by the recurrence relation

$$L_{k,0} = 2, L_{k,1} = k \text{ and } L_{k,n+1} = kL_{k,n} + L_{k,n-1}; n \geq 1.$$

These sequences were firstly studied by Horadam in [10]. The interested reader is also referred to [2, 6-8] for further details about these sequences.
The Cayley- Dickson algebras are real numbers, complex numbers, quaternions, octonions, sedenions and trigintaduonions. The Cayley- Dickson algebras have been studied in several papers.

In 1963, Horadam [11] introduced nth Fibonacci and nth Lucas quaternions. Many interesting properties of Fibonacci and Lucas quaternions have been presented in the literature (e.g. see [9,12, 15]). Octonions and sedenions that are hyper-complex
numbers as quaternions have been studied some of recent papers [1,13].
The trigintaduonions which are real algebras form a 32-dimensional the Cayley-Dickson algebra. A trigintaduonion is defined by

$$t = t_0 + \sum_{i=1}^{31} t_i e_i.$$

The trigintaduonions product is given in the matrix-vector multiplication form as: [3]

$$t_1 = a_0 + \sum_{i=1}^{31} a_i e_i, t_2 = b_0 + \sum_{i=1}^{31} b_i e_i, t_3 = t_1 t_2 = c_0 + \sum_{i=1}^{31} c_i e_i \quad (1)$$

The operations requiring for the matrix-vector multiplication in (1) are quite alot. Detailed informations about these operations have been presented in the literature (e.g. see [3-5,16]).
In this paper, we give some properties of k- Fibonacci and k-Lucas trigintaduonions and derive relationships between these trigintaduonions.

2. k- Fibonacci and k-Lucas Trigintaduonions

In this section, we introduce k-Fibonacci and k-Lucas trigintaduonions. Also, we give generating functions, Binet's formulas and some identities for these trigintaduonions.

Definition 1.
k-Fibonacci and k-Lucas trigintaduonions are defined by

$$TF_{k,n} = \sum_{s=0}^{31} F_{k,n+s} e_s \text{ and } TL_{k,n} = \sum_{s=0}^{31} L_{k,n+s} e_s$$

where $F_{k,n}$ and $L_{k,n}$ are the nth k-Fibonacci number and the nth k- Lucas number, respectively.
The sum of $TF_{k,n}$ and $TL_{k,n}$ is

$$TF_{k,n} \pm TL_{k,n} = \sum_{s=0}^{31} \left(F_{k,n+s} \pm L_{k,n+s}\right) e_s.$$

The conjugates of $TF_{k,n}$ and $TL_{k,n}$ are defined by, respectively,

$$TF^*{}_{k,n} = F_{k,n} e_0 - \sum_{s=1}^{31} F_{k,n+s} e_s \text{ and } TL^*{}_{k,n} = L_{k,n} e_0 - \sum_{s=1}^{31} L_{k,n+s} e_s.$$

The norm of $TF_{k,n}$ and $TL_{k,n}$ are defined by, respectively,

$$TF_{k,n} TF^*{}_{k,n} = \sum_{s=0}^{31} F_{k,n+s}^2 e_s \text{ and } TL_{k,n} TL^*{}_{k,n} = \sum_{s=0}^{31} L_{k,n+s}^2 e_s.$$

We can give some important identities of k- Fibonacci trigintaduonion and k-Lucas trigintaduonion as follows:

$$kTF_{k,n} + TF_{k,n-1} = TF_{k,n+1},$$
$$kTL_{k,n} + TL_{k,n-1} = TL_{k,n+1},$$
$$TF_{k,n-1} + TF_{k,n+1} = TL_{k,n}.$$

The characteristic equation of these sequences is $r^2 - kr - 1 = 0$. The roots of this equation are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$. Also, There are the following identities:

$$r_1 + r_2 = k,\ r_1 - r_2 = \sqrt{k^2+4},\ r_1 r_2 = -1.$$

Binet formulas for n-th $(n \geq 0)$ k-Fibonacci and k-Lucas numbers are given by

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}\ ,\ L_{k,n} = r_1^n + r_2^n,$$

respectively [6,8].

Theorem 2.
Binet's formula for $TF_{k,n}$ and $TL_{k,n}$, respectively, are given by

$$TF_{k,n} = \frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2} \text{ and } TL_{k,n} = \hat{r}_1 r_1^n - \hat{r}_2 r_2^n,$$

where $\hat{r}_1 = \sum_{i=0}^{31} r_1^i e_i$ and $\hat{r}_2 = \sum_{i=0}^{31} r_2^i e_i$.

Proof.
We know that $TF_{k,n} = Ar_1^n + Br_2^n$ (2)

For $n = 0$ and $n = 1$, $TF_{k,0} = A + B$ and $TF_{k,1} = Ar_1 + Br_2$ we obtain

$$A = \frac{TF_{k,1} - r_2 TF_{k,0}}{r_1 - r_2} = \frac{\hat{r}_1}{r_1 - r_2},$$
$$B = -\frac{TF_{k,1} - r_1 TF_{k,0}}{r_1 - r_2} = -\frac{\hat{r}_2}{r_1 - r_2}.$$

When these values write in equality (2), we have

$$TF_{k,n} = \frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}.$$

Similarly, we can see $TL_{k,n} = \hat{r}_1 r_1^n - \hat{r}_2 r_2^n$.

Using the Binet's formulas for $TF_{k,n}$ and $TL_{k,n}$, we give some relations between r_1, r_2, $\hat{r}_1, \hat{r}_2$ and k-Fibonacci or k- Lucas trigintaduonion by following lemma:

Lemma 3.

$$\hat{r}_1 r_1^n = TF_{k,n+1} - r_2 TF_{k,n} = \frac{TL_{k,n+1} - r_2 TL_{k,n}}{r_1 - r_2},$$
$$\hat{r}_2 r_2^n = TF_{k,n+1} - r_1 TF_{k,n} = \frac{TL_{k,n+1} - r_1 TL_{k,n}}{r_1 - r_2}.$$

Proof.

By applying Binet formula, we obtain the following equalities:

$$TF_{k,n+1} - r_2 TF_{k,n} = \frac{\hat{r}_1 r_1^{n+1} - \hat{r}_2 r_2^{n+1}}{r_1 - r_2} - r_2 \frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}$$
$$= \frac{1}{r_1 - r_2}(\hat{r}_1 r_1^{n+1} - \hat{r}_2 r_2^{n+1} - \hat{r}_1 r_2 r_1^n + \hat{r}_2 r_2^{n+1}) = \hat{r}_1 r_1^n.$$
$$TF_{k,n+1} - r_1 TF_{k,n} = \frac{\hat{r}_1 r_1^{n+1} - \hat{r}_2 r_2^{n+1}}{r_1 - r_2} - r_1 \frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}$$
$$= \frac{1}{r_1 - r_2}(\hat{r}_1 r_1^{n+1} - \hat{r}_2 r_2^{n+1} + \hat{r}_2 r_1 r_2^n - \hat{r}_1 r_1^{n+1}) = \hat{r}_2 r_2^n.$$

Theorem 4.

(Catalan's identity) For $n \geq r \geq 1$, we have the following formulas:

$$TF_{k,n-r} TF_{k,n+r} - TF_{k,n}^2 = \frac{(-1)^{n-r+1}}{k^2+4}\big((\hat{r}_1 \hat{r}_2)(r_2^{2r} - (-1)^r) + (\hat{r}_2 \hat{r}_1)(r_1^{2r} - (-1)^r)\big),$$
$$TL_{k,n-r} TL_{k,n+r} - TL_{k,n}^2 = (-1)^{n-r+1}\big((\hat{r}_1 \hat{r}_2)(r_2^{2r} - (-1)^r) + (\hat{r}_2 \hat{r}_1)(r_1^{2r} - (-1)^r)\big).$$

Proof.

By using Binet formula and $r_1 r_2 = -1$, we obtain

$$TF_{k,n-r} TF_{k,n+r} - TF_{k,n}^2 = \frac{\hat{r}_1 r_1^{n-r} - \hat{r}_2 r_2^{n-r}}{r_1 - r_2} \frac{\hat{r}_1 r_1^{n+r} - \hat{r}_2 r_2^{n+r}}{r_1 - r_2} - \left(\frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}\right)^2$$

$$= \frac{\hat{r}_1 \hat{r}_2(-r_1^{n-r} r_2^{n+r} + r_1^n r_2^n) + \hat{r}_2 \hat{r}_1(-r_2^{n-r} r_1^{n+r} + r_1^n r_2^n)}{(r_1 - r_2)^2}$$

$$= \frac{(-1)^{n-r+1}\big((\hat{r}_1 \hat{r}_2)(r_2^{2r} - (-1)^r) + (\hat{r}_2 \hat{r}_1)(r_1^{2r} - (-1)^r)\big)}{(r_1 - r_2)^2}$$
$$= \frac{(-1)^{n-r+1}}{k^2+4}\big((\hat{r}_1 \hat{r}_2)(r_2^{2r} - (-1)^r) + (\hat{r}_2 \hat{r}_1)(r_1^{2r} - (-1)^r)\big)$$

$$TL_{k,n-r} TL_{k,n+r} - TL_{k,n}^2$$
$$= (\hat{r}_1 r_1^{n-r} - \hat{r}_2 r_2^{n-r})(\hat{r}_1 r_1^{n+r} - \hat{r}_2 r_2^{n+r}) - (\hat{r}_1 r_1^n - \hat{r}_2 r_2^n)^2$$
$$= (-1)^{n-r+1}\big((\hat{r}_1 \hat{r}_2)(r_2^{2r} - (-1)^r) + (\hat{r}_2 \hat{r}_1)(r_1^{2r} - (-1)^r)\big).$$

The Cassini's identity given in the following theorem is the special case of Catalan's identity.

Theorem 5.
(Cassini's identity) For $n \geq 1$, we have the following formulas:

$$TF_{k,n-1}TF_{k,n+1} - TF_{k,n}^2 = \frac{(-1)^n}{k^2+4}\left((\hat{r}_1\hat{r}_2)(r_2^2+1) + (\hat{r}_2\hat{r}_1)(r_1^2+1)\right),$$

$$TL_{k,n-1}TL_{k,n+1} - TL_{k,n}^2 = (-1)^n\left((\hat{r}_1\hat{r}_2)(r_2^2+1) + (\hat{r}_2\hat{r}_1)(r_1^2+1)\right).$$

Theorem 6.
If $m > n$, then we get

$$TF_{k,m}TF_{k,n+r} - TF_{k,m+r}TF_{k,n} = \frac{(-1)^n}{\sqrt{k^2+4}}F_{k,r}(\hat{r}_1\hat{r}_2r_1^{m-n} - \hat{r}_2\hat{r}_1r_2^{m-n}),$$

$$TL_{k,m}TL_{k,n+r} - TL_{k,m+r}TL_{k,n} = (-1)^n\sqrt{k^2+4}F_{k,r}(\hat{r}_1\hat{r}_2r_1^{m-n} - \hat{r}_2\hat{r}_1r_2^{m-n}).$$

Proof.
By using Binet formula and $r_1r_2 = -1$, we obtain

$$TF_{k,m}TF_{k,n+\mathrm{r}} - TF_{k,m+\mathrm{r}}TF_{k,n} = \left(\frac{\hat{r}_1r_1^m - \hat{r}_2r_2^m}{r_1-r_2}\frac{\hat{r}_1r_1^{n+\mathrm{r}} - \hat{r}_2r_2^{n+\mathrm{r}}}{r_1-r_2}\right) - \left(\frac{\hat{r}_1r_1^{m+\mathrm{r}} - \hat{r}_2r_2^{m+\mathrm{r}}}{r_1-r_2}\frac{\hat{r}_1r_1^n - \hat{r}_2r_2^n}{r_1-r_2}\right)$$

$$= \frac{(-1)^n}{\sqrt{k^2+4}}F_{k,r}(\hat{r}_1\hat{r}_2r_1^{m-n} - \hat{r}_2\hat{r}_1r_2^{m-n}).$$

The second identity is found in a similar manner.

Corollary 7.
If $m > n$, then we get

$$TF_{k,m}TF_{k,n+1} - TF_{k,m+1}TF_{k,n} = \frac{(-1)^n}{\sqrt{k^2+4}}(\hat{r}_1\hat{r}_2r_1^{m-n} - \hat{r}_2\hat{r}_1r_2^{m-n}),$$

$$TL_{k,m}TL_{k,n+1} - TL_{k,m+1}TL_{k,n} = (-1)^n\sqrt{k^2+4}(\hat{r}_1\hat{r}_2r_1^{m-n} - \hat{r}_2\hat{r}_1r_2^{m-n}).$$

Proof.
Taking $r = 1$ a special case of theorem 6, the proof can easily seen.

Theorem 8.
The relationship between $TF_{k,n}$ and $TL_{k,n}$ is as follows:

i. $TF_{k,n}^2 = \frac{1}{k^2+4}\mathrm{T}L^2{}_{k,n}$,

ii. $TL_{k,n}^2 + TF_{k,n}^2 = \frac{k^2+5(\mathrm{T}L^2{}_{k,n})}{k^2+4}$

Proof. By using Binet formula for $TF_{k,n}$ and $TL_{k,n}$, we have

i. $$TL_{k,n}^2 - TF_{k,n}^2 = (\hat{r}_1 r_1^n - \hat{r}_2 r_2^n)^2 - \left(\frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}\right)^2$$
$$= ((r_1 - r_2)^2 - 1)\frac{(\hat{r}_1 r_1^n - \hat{r}_2 r_2^n)^2}{(r_1 - r_2)^2}$$
$$= \frac{k^2 + 3(\mathrm{T}L^2{}_{k,n})}{k^2+4}$$
$$TF_{k,n}^2 = \frac{1}{k^2+4}\mathrm{T}L^2{}_{k,n}.$$

ii. $$TL_{k,n}^2 + TF_{k,n}^2 = (\hat{r}_1 r_1^n - \hat{r}_2 r_2^n)^2 + \left(\frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}\right)^2$$
$$= \frac{k^2 + 5(\mathrm{T}L^2{}_{k,n})}{k^2+4}.$$

Theorem 9.
The generating functions for the k-Fibonacci and k-Lucas trigintaduonions are

$$\sum_{i=0}^{\infty} TF_{k,i}x^i = \frac{TF_{k,0} + TF_{k,-1}x}{1 - kx - x^2}$$

and

$$\sum_{i=0}^{\infty} TL_{k,i}x^i = \frac{TL_{k,0} + TL_{k,-1}x}{1 - kx - x^2}.$$

Proof.
Define $G(TF_{k,i};x) = \sum_{i=0}^{\infty} TF_{k,i}x^i$. Multiplying both sides of this equation by $-kx$ and $-x^2$, we obtain the following equations;

$$G(TF_{k,i};x) = TF_{k,0} + TF_{k,1}x + TF_{k,2}x^2 + \cdots + TF_{k,i}x^i + \cdots,$$
$$-kxG(TF_{k,i};x) = -kTF_{k,0}x - kTF_{k,1}x^2 - kTF_{k,2}x^3 - \cdots - kTF_{k,i}x^{i+1} - \cdots,$$
$$-x^2G(TF_{k,i};x) = -TF_{k,0}x^2 - TF_{k,1}x^3 - TF_{k,2}x^4 - \cdots - TF_{k,i}x^{i+2} - \cdots.$$

By adding these equations and using the identity $kTF_{k,n} + TF_{k,n-1} = TF_{k,n+1}$, we obtain

$$G(TF_{k,i};x) = \frac{TF_{k,0} + TF_{k,1}x - kTF_{k,0}x}{1 - kx - x^2}$$
$$= \frac{TF_{k,0} + TF_{k,-1}x}{1 - kx - x^2}.$$

The generating function for the k-Lucas trigintaduonion is obtained in a similar way.

Theorem 10.
For any integer m, n, the generating functions of the k-Fibonacci trigintaduonion $TF_{k,m+n}$ and the k-Lucas $TL_{k,m+n}$ trigintaduonion are

$$\sum_{n=0}^{\infty} TF_{k,m+n}\, x^n = \frac{TF_{k,m} + xTF_{k,m-1}}{1 - kx - x^2}$$

and

$$\sum_{n=0}^{\infty} TL_{k,m+n}\, x^n = \frac{TL_{k,m} + xTL_{k,m-1}}{1 - kx - x^2}.$$

Proof.

$$\sum_{n=0}^{\infty} TF_{k,m+n}\, x^n = \sum_{n=0}^{\infty} \left(\frac{\hat{r}_1 r_1^{m+n} - \hat{r}_2 r_2^{m+n}}{r_1 - r_2}\right) x^n$$

$$= \frac{\hat{r}_1 r_1^m}{r_1 - r_2} \sum_{n=0}^{\infty} r_1^n x^n - \frac{\hat{r}_2 r_2^m}{r_1 - r_2} \sum_{n=0}^{\infty} r_2^n x^n$$

$$= \frac{\hat{r}_1 r_1^m}{r_1 - r_2} \frac{1}{1 - xr_1} - \frac{\hat{r}_2 r_2^m}{r_1 - r_2} \frac{1}{1 - xr_2}$$

$$= \frac{1}{r_1 - r_2} \left(\frac{(\hat{r}_1 r_1^m - \hat{r}_2 r_2^m) + x(\hat{r}_1 r_1^{m-1} - \hat{r}_2 r_2^{m-1})}{1 - kx - x^2} \right)$$

$$= \frac{TF_{k,m} + xTF_{k,m-1}}{1 - kx - x^2}.$$

The proof of the second sum is found in a similar manner.

Theorem 11.

For any integer n, we have

$$\sum_{i=0}^{n} \binom{n}{i} k^i TF_{k,i} = TF_{k,2n},$$

$$\sum_{i=0}^{n} \binom{n}{i} k^i TL_{k,i} = TL_{k,2n}.$$

Proof.

$$\sum_{i=0}^{n} \binom{n}{i} k^i TF_{k,i} = \sum_{i=0}^{n} \binom{n}{i} k^i \frac{\hat{r}_1 r_1^i - \hat{r}_2 r_2^i}{r_1 - r_2}$$

$$= \frac{\hat{r}_1}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} k^i\, r_1^i - \frac{\hat{r}_2}{r_1 - r_2} \sum_{i=0}^{n} \binom{n}{i} k^i\, r_2^i$$

$$= \frac{\hat{r}_1}{r_1 - r_2} (1 + kr_1)^n - \frac{\hat{r}_2}{r_1 - r_2} (1 + kr_2)^n$$

r_1 and r_2 are roots of the equation $r^2 - kr - 1 = 0$. Thus, we can write $1 + kr_1 = (r_1)^2$ and $1 + kr_2 = (r_2)^2$. Using these equations in the last equation, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} k^i TF_{k,i} = \frac{\hat{r}_1 r_1^{2n} - \hat{r}_2 r_2^{2n}}{r_1 - r_2} = TF_{k,2n}.$$

The second identity is found in the same manner.

Now, we give the matrix representations of k -Fibonacci and k -Lucas trigintaduonions.

Theorem 12.

Let $n \geq 1$ be integer. Then we have

$$\begin{bmatrix} TF_{k,n} & TF_{k,n-1} \\ TF_{k,n+1} & TF_{k,n} \end{bmatrix} = \begin{bmatrix} TF_{k,1} & TF_{k,0} \\ TF_{k,2} & TF_{k,1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^{n-1},$$

$$\begin{bmatrix} TL_{k,n} & TL_{k,n-1} \\ TL_{k,n+1} & TL_{k,n} \end{bmatrix} = \begin{bmatrix} TL_{k,1} & TL_{k,0} \\ TL_{k,2} & TL_{k,1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

Proof.

The proof will be done by induction steps. Firstly, for $n = 1$, it holds the equation. Assume that the equation holds for all n, that is,

$$\begin{bmatrix} TF_{k,n} & TF_{k,n-1} \\ TF_{k,n+1} & TF_{k,n} \end{bmatrix} = \begin{bmatrix} TF_{k,1} & TF_{k,0} \\ TF_{k,2} & TF_{k,1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^{n-1},$$

We can end up the proof if we show that

$$\begin{bmatrix} TF_{k,n+1} & TF_{k,n} \\ TF_{k,n+2} & TF_{k,n+1} \end{bmatrix} = \begin{bmatrix} TF_{k,1} & TF_{k,0} \\ TF_{k,2} & TF_{k,1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^{n}.$$

By using induction's hypothesis, we have

$$\begin{aligned} \begin{bmatrix} TF_{k,1} & TF_{k,0} \\ TF_{k,2} & TF_{k,1} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} TF_{k,n} & TF_{k,n-1} \\ TF_{k,n+1} & TF_{k,n} \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} kTF_{k,n} + TF_{k,n-1} & TF_{k,n} \\ kTF_{k,n+1} + TF_{k,n} & TF_{k,n+1} \end{bmatrix} \\ &= \begin{bmatrix} TF_{k,n+1} & TF_{k,n} \\ TF_{k,n+2} & TF_{k,n+1} \end{bmatrix}. \end{aligned}$$

Thus, the proof is ended.

The remaining part of the theorem is proved in a similar way.

Corollary 13.

$$TF_{k,n-1}TF_{k,n+1} - TF_{k,n}^2 = (-1)^{n-1}\left(TF_{k,0}TF_{k,2} - TF_{k,1}^2\right),$$

$$TL_{k,n-1}TL_{k,n+1} - TL_{k,n}^2 = (-1)^{n-1}\left(TL_{k,0}TL_{k,2} - TL_{k,1}^2\right).$$

Proof.
If there exist the determinations of the matrix representations of $TF_{k,n}$ and $TL_{k,n}$, the proof can be easily seen.

References

[1] G. Bilgici, Ü. Tokeşer and Z. Ünal, Fibonacci and Lucas Sedenions, *Journal of Integer Sequences*, **20** (2017), 1-11.

[2] C. Bolat and H. Köse, On the properties of k-Fibonacci numbers, *Int. J. Contemp. Math. Sciences*, **5** (2010), no. 22, 1097-1105.

[3] A. Cariow and G. Cariowa, An algorithm for multiplication of trigintaduonions, *Journal of Theoretical and Applied Computer Science*, **8** (2014), no. 1, 50-75.

[4] R.E. Cawagas, A.S. Carrascal, L.A. Bautista, J.P. Sta. Maria, J.D. Urrutia, B. Nobles, The Subalgebra Structure of the Cayley-Dickson Algebra of Dimension 32 (Trigintaduonions), arXiv:0907.2047v3, (2009).

[5] W.L. Chan, H. Choi and R.G. Baraniuk, Directional hypercomplex wavelets for multidimensional signal analysis and processing, *2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, (2004), 996-999. https://doi.org/10.1109/icassp.2004.1326715

[6] S. Falcon, On the k-Lucas numbers, *Int. J. Contemp. Math. Sciences*, **6** (2011), no. 21, 1039-1050.

[7] S. Falcon, The k-Fibonacci matrix and the Pascal matrix, *Cent. Eur. J. Math.*, **9** (2011), no. 6, 1403-1410. https://doi.org/10.2478/s11533-011-0089-9

[8] S. Falcon and A. Plaza, On the Fibonacci k-numbers, *Chaos Solitons Fractals*, **32** (2007), no. 5, 1615-1624. https://doi.org/10.1016/j.chaos.2006.09.022

[9] S. Halici, On Fibonacci quaternions, *Adv. Appl. Clifford Algebras*, **22** (2012), 321–327. https://doi.org/10.1007/s00006-011-0317-1

[10] A.F. Horadam, A generalized Fibonacci sequence, *Amer. Math. Montly*, **68** (1961), 455-459. https://doi.org/10.2307/2311099

[11] A.F. Horadam, Complex Fibonacci numbers and Fibonacci quaternions, *Am. Math. Month.*, **70** (1963), 289–291. https://doi.org/10.2307/2313129

[12] M.R. Iyer, A note on Fibonacci quaternions, *Fibonacci Quaterly*, **3** (1969), 225–229.

[13] O. Keçilioğlu and I. Akkus, The Fibonacci Octonions, *Adv. Appl. Clifford Algebras*, **25** (2015), no. 1, 151–158. https://doi.org/10.1007/s00006-014-0468-y

[14] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Interscience Publication, 2001. https://doi.org/10.1002/9781118033067

[15] J.L. Ramirez, Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions, *An. St. Univ. Ovidus Constanta*, **23** (2015), no. 2. https://doi.org/10.1515/auom-2015-0037

[16] Z.H. Weng, Compounding Fields and Their Quantum Equations in the Trigintaduonion Space, arXiv: 0704.0136.

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