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Almost Contra θ-C-Continuous Functions

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Abstract

A weak form of contra θ-c-continuity, called almost contra θ-c-continuity is introduced. Also a local form of almost contra θ-c-continuity is investigated. The basic properties of both of these classes of functions are developed and an application to H-closed spaces and Katětov spaces is investigated.

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1 Introduction

Contra-continuity was introduced by Dontchev [6] in 1996. Since then many variations of contra continuity have been studied. Recently several different forms of almost contra continuity have been investigated. Ekici [9, 10] developed the notions of an almost contra-super-continuous function and an almost contra precontinuous function. Recently Caldas, et al. [4] have studied almost contra $\beta\theta$-continuity [4]. The purpose of this note is to introduce the concept of almost contra θ-c-continuity, which is a weak form of contra θ-c-continuity, introduced by Baker [3]. The class of almost contra θ-c-continuous functions fills gaps between several classes of functions and the class of almost contra-super-continuous functions. Finally we develop a local version of almost contra θ-c-continuity, which we call almost local contra θ-c-continuity. This is a weak form of local contra θ-c-continuity, introduced by Baker [3]. The basic properties of both of these classes of functions are developed. For example, conditions are established under which the range of an almost locally contra θ-c-continuous function is nearly compact. An application of almost locally contra θ-c-continuity to H-closed spaces and Katětov spaces is investigated.

2 Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a subset A of a space X are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is said to be regular open provided that $A = \mathrm{Int}(\mathrm{Cl}(A))$ and regular closed provided that $A = \mathrm{Cl}(\mathrm{Int}(A))$ or equivalently its complement is regular open. The θ-closure of a set A, denoted by $\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ such that every closed neighborhood of x intersects A nontrivially. A set A is said to be θ-closed [13] if $\mathrm{Cl}_\theta(A) = A$. A set A is θ-open if its complement is θ-closed or equivalently if A contains a closed neighborhood of each of its points. A set A is called δ-open [13] if A contains a regular open neighborhood of each of its points and δ-closed if its complement is δ-open. A subset A of a space X is θ-c-open [2] if there exists a set B such that $A = X - \mathrm{Cl}_\theta(B)$ and θ-c-closed if its complement is θ-c-open or equivalently if there exists a set B such that $A = \mathrm{Cl}_\theta(B)$.

Definition 2.1 *A function $f : X \to Y$ is said to be contra θ-c-continuous* [3] *if $f^{-1}(V)$ is θ-c-closed for every open set V of Y.*

Definition 2.2 *A function $f : X \to Y$ is said to be almost contra supercontinuous* [9] *(respectively, almost contra θ-continuous) if $f^{-1}(V)$ is δ-closed (respectively, θ-closed) for every regular open set V of Y.*

Definition 2.3 *A function $f : X \to Y$ is said to be RC-continuous* [7] *if $f^{-1}(V)$ is regular closed for every open set V of Y.*

Definition 2.4 *A function $f : X \to Y$ is said to be an R-map* [5] *(respectively, a contra R-map* [8]*) if $f^{-1}(V)$ is regular open (respectively, regular closed) for every regular open set V of Y.*

Definition 2.5 *A space X is said to be H-closed* [11] *if X is a closed subset in every space containing X as a subspace.*

Definition 2.6 *A space X is said to be Katětov* [11] *if it has a coarser minimal H-closed topology or equivalently a coarser H-closed topology.*

3 Almost contra θ-c-continuous functions

Definition 3.1 *A function $f : X \to Y$ is said to be almost contra θ-c-continuous if $f^{-1}(V)$ is θ-c-closed for every regular open subset V of Y.*

Obviously contra θ-c-continuity implies almost contra θ-c-continuity. Since regular closed implies θ-c-closed [2], contra R-map implies almost contra θ-c-continuity. Since θ-c-open implies δ-open [2], almost contra θ-c-continuity implies almost contra-super-continuity. Since θ-open implies θ-c-open [2], almost contra θ-continuity implies almost contra θ-c-continuity.

Thus we have the following diagram of implications.

contra R-map
$\Downarrow$
almost contra θ-cont. $\Rightarrow$ almost contra-θ-c-cont. $\Rightarrow$ almost contra-super-cont.
$\Uparrow$
contra θ-c-cont.

The following examples show that none of these implications are reversible.

Example 3.2 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\} \{b\}, \{a, b\}\}$ and let $f : (X, \tau) \to (Y, \tau)$ be given by $f(a) = a, f(b) = c,$ and $f(c) = a$. Then f is almost contra θ-c-continuous, but, since $f^{-1}(\{a\})$ is not θ-closed, f is not almost contra θ-continuous.*

Example 3.3 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ and let $f : (X, \tau) \to (Y, \tau)$ be the identity map. Then f is obviously almost contra-super-continuous, but, since $f^{-1}(\{a\})$ is not θ-c-closed, f is not almost contra θ-c-continuous.*

Example 3.4 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\} \{b\}, \{a, b\}\}$ and let $f : (X, \tau) \to (Y, \tau)$ be given by $f(a) = c, f(b) = c,$ and $f(c) = a$. Then f is almost contra-super-continuous, but, since $f^{-1}(\{a\})$ is not θ-c-closed, f is not almost contra θ-c-continuous.*

Example 3.5 *Let X denote the real numbers, let τ be the usual topology on X, and let $\sigma = \{U \subseteq X : 1 \notin U \text{ or } U = X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be given by $f(x) = 2$ if $x \leq 0$ or $x = 1$ and $f(x) = 1$ if $0 < x < 1$ or $x > 1$. Then f is almost contra θ-c-continuous, but not a contra R-map.*

The next two results are consequences of the definition of a θ-c-closed set.

Theorem 3.6 *If $f : X \to Y$ is almost contra θ-c-continuous and $Cl_\theta(A)$ is regular-closed for every set A in X, then f is a contra-R-map.*

Theorem 3.7 *If $f : X \to Y$ is almost contra θ-c-continuous and $Cl_\theta(A)$ is θ-closed for every set A in X, then f is almost contra θ-continuous.*

Definition 3.8 *A function $f : X \to Y$ is said to be quasi θ-c-continuous* [3] *if $f^{-1}(V)$ is θ-c-open for every θ-c-open subset V of Y.*

Theorem 3.9 *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*

(a) *If f is quasi-θ-c-continuous and g is almost contra θ-c-continuous, then $g \circ f$ is almost contra θ-c-continuous.*

(b) *If f is almost contra θ-c-continuous and g is an R-map, then $g \circ f$ is almost contra θ-c-continuous.*

Definition 3.10 *A function $f : X \to Y$ is said to be quasi θ-c-closed if $f(F)$ is θ-c-closed for θ-c-closed set F in X.*

Theorem 3.11 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ is almost contra θ-c-continuous and f is surjective and quasi θ-c-closed, then g is almost contra θ-c-continuous..*

Proof. Let V be a regular open set in Z. Then $f^{-1}(g^{-1}(V))$ is θ-c-closed in X. Thus $f(f^{-1}(g^{-1}(V)))$ is θ-c-closed in Y. Since f is surjective, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$. Hence $g^{-1}(V)$ is θ-c-closed in Y, which proves that g is almost contra θ-c-continuous.

Definition 3.12 *A function $f : X \to Y$ is said to be quasi regular-open if $f(V)$ is regular open for every regular open set V in X.*

Theorem 3.13 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ is almost contra θ-c-continuous and g is injective and quasi regular open, then f is almost contra θ-c-continuous..*

Proof Let V be a regular open set in Y. Since g is quasi regular open, $g(V)$ is regular open in Z. Then, since $g \circ f$ is almost contra θ-c-continuous $f^{-1}(V) = f^{-1}(g^{-1}(g(V)))$ is θ-c-closed in X. Thus f is almost contra θ-c-continuous.

4 Almost locally contra θ-c-continuous functions

Definition 4.1 *A function $f : X \to Y$ is said to be almost locally contra θ-c-continuous (respectively, locally contra θ-c-continuous* [3]*) if for every $x \in X$ and every regular open (respectively, open) set V in Y containing $f(x)$, there exists a θ-c-closed set F in X such that $x \in F$ and $f(F) \subseteq V$.*

It follows from the definitions that almost locally contra θ-c-continuity is implied by both almost contra θ-c-continuity and locally contra θ-c-continuity. The following examples show that these implications are not reversible,

Example 4.2 *Let X denote the real numbers with the usual topology. The identity mapping $f : X \to X$ is almost locally contra θ-c-continuous, since for any nonempty set V, $f^{-1}(V)$ is a union of singleton sets and hence a union of θ-c-closed sets. However, f is not almost contra θ-c-continuous, because $(0, 1)$ is regular open, but not θ-c-closed.*

The function in Example 3.3 is almost locally contra θ-c-continuous, but not locally contra θ-c-continuous. Incidentally, as the next example shows, almost locally contra θ-c-continuity does not imply almost contra-super-continuity.

Example 4.3 *Let X denote the real number and let σ be the usual topology on X and let τ be the discrete topology on X. The identity mapping $f : (X, \sigma) \to (X, \tau)$ is almost locally contra θ-c-continuous but not almost contra-super-continuous. Note that any singleton set is regular closed in (X, τ) but not δ-open in (X, σ).*

Definition 4.4 *A space X is said to be rT_1* [1] *if for every pair of distinct points x and y of X there exists δ-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.*

Theorem 4.5 *If $f : X \to Y$ is an almost locally contra θ-c-continuous injection and Y is Urysohn, then X is rT_1.*

Proof. Let x and y be distinct points in X. Since Y is Urysohn, there exist open sets V and W containing $f(x)$ and $f(y)$, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since $\mathrm{Cl}(V)$ and $\mathrm{Cl}(W)$ are regular closed, $f^{-1}(\mathrm{Cl}(V))$ and $f^{-1}(\mathrm{Cl}(W))$ are intersections of θ-c-open sets. Thus there exist $\{A_\alpha : \alpha \in \mathcal{A}\}$ and $\{B_\beta : \beta \in \mathcal{B}\}$ such that A_α is θ-c-open for every $\alpha \in \mathcal{A}$ and B_β is θ-c-open for every $\beta \in \mathcal{B}$ and $f^{-1}(\mathrm{Cl}(V)) = \cap_{\alpha \in \mathcal{A}} A_\alpha$ and $f^{-1}(\mathrm{Cl}(W)) = \cap_{\beta \in \mathcal{B}} B_\beta$. Since $f^{-1}(\mathrm{Cl}(V)) \cap f^{-1}(\mathrm{Cl}(W)) = \emptyset$, there exists $\alpha \in \mathcal{A}$ such that $x \in A_\alpha$ and $y \notin A_\alpha$ and there exists $\beta \in \mathcal{B}$ such that $y \in B_\beta$ and $x \notin B_\beta$. Since θ-c-open implies δ-open, A_α and B_β are δ-open. Thus X is rT_1.

Definition 4.6 *A space X is said to be strongly θ-c-closed* [3] *if every cover of X by θ-c-closed sets has a finite subcover.*

Definition 4.7 *A space X is said to be nearly compact* [12] *if every cover of X by regular open sets has a finite subcover.*

Theorem 4.8 *If $f : X \to Y$ is almost locally contra θ-c-continuous and surjective and X is strongly θ-c-closed, then Y is nearly compact.*

Proof. Let $\mathcal{C}$ be a cover of Y by regular open sets. Let $x \in X$ and let $V_x \in \mathcal{C}$ such that $f(x) \in V_x$. Then, since f is almost locally contra θ-c-continuous, there exists a θ-c-closed set F_x such that $x \in F_x \subseteq f^{-1}(V_x)$. Then, since $\{F_x : x \in X\}$ is a cover of X by θ-c-closed sets, there exists a finite subcover $\{F_{x_i} : i = 1, \ldots, n\}$. It then follows that $\{V_{x_i} : i = 1, \ldots, n\}$ is a finite subcover of $\mathcal{C}$, which proves that Y is nearly compact.

Recall that the graph of a function $f : X \to Y$ is given by $G(f) = \{(x, f(x)) : x \in X\}$.

Definition 4.9 *The graph of a function $f : X \to Y$ is said to be almost contra θ-c-closed if for every $(x, y) \in X \times Y - G(f)$, there exist a θ-c-closed set F in X and a regular open set V in Y such that $(x, y) \in F \times V \subseteq X \times Y - G(f)$.*

Theorem 4.10 *If $f : X \to Y$ is almost locally contra θ-c-continuous and Y is Hausdorff, then $G(f)$ is almost contra θ-c-closed.*

Proof Let $(x, y) \in X \times Y - G(f)$. Then, since $y \neq f(x)$, there exist disjoint open sets V and W such that $f(x) \in V$ and $y \in W$. Since $Y - \mathrm{Cl}(W)$ is regular open and f is almost locally contra θ-c-continuous, there exists a θ-c-closed set F such that $x \in F \subseteq f^{-1}(Y - \mathrm{Cl}(W))$. Then we see that $(x, y) \in F \times \mathrm{Int}(\mathrm{Cl}(W)) \subseteq X \times Y - G(f)$, which proves that $G(f)$ is almost contra θ-c-closed.

Lemma 4.11 *The graph of a function $f : X \to Y$ is almost contra θ-c-closed if and only if for every $(x, y) \in X \times Y - G(f)$ there exists a θ-c-closed set F containing x and a regular open set V containing y such that $f(F) \cap V = \emptyset$.*

Theorem 4.12 *If $f : X \to Y$ has an almost contra-θ-closed graph, then for every $x \in X$, $\{f(x)\} = \cap\{Cl(f(F)) : F$ is θ-c-closed and $x \in F\}$.*

Proof Assume the statement is false. Then for some $x \in X$ there exists $y \in Y$ such that $y \neq f(x)$ and $y \in \mathrm{Cl}(f(F))$ for every θ-c-closed set F containing x. So for every open set V in Y containing y and every θ-c-closed set F containing x, $V \cap f(F) \neq \emptyset$. This contradicts the fact that $G(f)$ is almost contra θ-c-closed.

Corollary 4.13 *If $f : X \to Y$ has an almost contra-θ-closed graph, then $\{f(x)\}$ is closed for every x in X.*

Theorem 4.14 *If $f, g : X \to Y$ are almost locally contra θ-c-continuous and Y is Hausdorff, then the set $A = \{x : f(x) \neq g(x)\}$ is the union of intersections of pairs of θ-c-closed sets.*

Proof. Let $x \in A$. Since Y is Hausdorff, there exist disjoint open sets V and W containing $f(x)$ and $g(x)$, respectively. Then $f(x) \in \text{Int}(\text{Cl}(V))$ and $g(x) \in \text{Int}(\text{Cl}(W))$ and $\text{Int}(\text{Cl}(V)) \cap \text{Int}(\text{Cl}(W)) = \emptyset$. Since f and g are almost locally contra θ-c-continuous, there exist θ-c-closed sets F and G such that $x \in F \subseteq f^{-1}(\text{Int}(\text{Cl}(V)))$ and $x \in G \subseteq g^{-1}(\text{Int}(\text{Cl}(W)))$. Then $x \in F \cap G \subseteq A$, which proves that A is the union of intersections of pairs of θ-c-closed sets.

Theorem 4.15 [11] *If X is an H-closed space and $A \subseteq X$, then $Cl_\theta(A)$ is Katětov.*

It is an immediate consequence of Theorem 4.15 that every θ-c-closed subset of an H-closed space is Katětov. Thus the following result is a consequence of the definition of almost local contra θ-c-continuity.

Theorem 4.16 *If $f : X \to Y$ is almost locally contra θ-c-continuous and X is H-closed, then for every regular open set V in Y, $f^{-1}(V)$ is a union of Katětov spaces.*

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