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Almost Contra-Somewhat Continuity

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Abstract

Two weak forms of contra-somewhat continuity, called almost contra-1-somewhat continuity and almost contra-2-somewhat continuity are introduced. It is shown that each of these forms is weaker than the corresponding version of contra-somewhat continuity. The basic properties of these functions are developed and relationships between these forms and other generalized continuity conditions are investigated.

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1 Introduction

The class of somewhat continuous functions was studied by Gentry and Hoyle [8] in 1971. Two types of contra-somewhat continuity were developed in 2015 by Baker [1]. The purpose of this note is to introduce weak forms of both of these types of contra somewhat continuity, which we call almost contra-1-somewhat continuity and almost contra-2-somewhat continuity. It is established that almost contra-1-somewhat continuity and almost contra-2-somewhat continuity are independent of each other and strictly weaker than contra-1-somewhat continuity and contra-2-somewhat continuity, respectively.

Almost contra-1-somewhat continuity appears to be the more interesting of the two forms, since almost contra-2-somewhat continuity turns out to be equivalent to contra-2-somewhat continuity when the topology on the codomain is modified. Characterizations and the basic properties are developed. Almost contra-1-somewhat continuity is characterized by mapping dense sets to sets with large regular kernels and almost contra-2-somewhat continuity is characterized by mapping sets with large kernels to δ-dense sets.

2 Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is said to be preopen [11] (respectively, semiopen [10]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, (respectively, $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$). A set A is preclosed (respectively, semi-closed provided its complement is preopen (respectively, semi-open). A set A is regular open (respectively, regular closed) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (respectively, $\mathrm{Cl}(\mathrm{Int}(A)) = A$). A set A is called δ-open [15] if for each $x \in A$ there exists a regular open set U such that $x \in U \subseteq A$. The family of all δ-open sets in a space (X, τ) is a topology on X and is denoted by τ_δ. (This topology is also referred to as the semi-regularization topology and denoted by τ_s.) The collection of all regular open sets forms a base for τ_δ and the space (X, τ_δ) will be denoted by X_δ.

Definition 2.1 *A function $f : X \to Y$ is said to be contra-continuous* [3] *(respectively, contra-almost continuous* [2] *if $f^{-1}(V)$ is closed for every open (respectively, regular open)subset V of Y.*

Definition 2.2 *A function $f : X \to Y$ is said to be almost contra-precontinuous* [7] *(respectively, almost contra-semicontinuous) if $f^{-1}(V)$ is preclosed (respectively, semi-closed) for every regular open subset V of Y.*

Definition 2.3 *A function $f : X \to Y$ is said to be somewhat continuous* [8]*, if for every open subset V of Y such that $f^{-1}(V) \neq \emptyset$, there exists an open subset U of X such that $\emptyset \neq U \subseteq f^{-1}(V)$.*

Definition 2.4 *A function $f : X \to Y$ is said to be contra-1-somewhat continuous* [1] *provided that for every closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists an open set $U \subseteq X$ such that $\emptyset \neq U \subseteq f^{-1}(F)$.*

Definition 2.5 *A function $f : X \to Y$ is said to be contra-2-somewhat continuous* [1] *if for every open set $V \subset Y$ such that $f^{-1}(V) \neq \emptyset$, there exists a closed set $F \subseteq X$ such that $\emptyset \neq F \subseteq f^{-1}(V)$.*

Definition 2.6 *A function $f : X \to Y$ is said to be an R-map* [5] *if $f^{-1}(V)$ is regular open for every regular open subset V of Y.*

Definition 2.7 *A function $f : X \to Y$ is said to be almost semicontinuous* [13] *if $f^{-1}(V)$ is semiopen for every regular open subset V of Y.*

Definition 2.8 *Let A be a subset of a space X. The kernel (respectively r-kernel* [6]*) of A* [12]*, denoted by ker(A), (respectively, r-ker(A)), is the intersection of all open (respectively, regular open) subsets of X containing A.*

Lemma 2.9 [9] *The following statements hold for subsets A and B of a space X:*

(a) *$x \in$ ker(A) if and only if $A \cap F \neq \emptyset$ for every closed subset F of X containing x.*

(b) *$A \subseteq ker(A)$ and $A = ker(A)$ if A is open in X.*

(c) *If $A \subseteq B$, then $ker(A) \subseteq ker(B)$.*

Remark 2.10 *The analogous properties (see Lemma 1* [6]*) hold for the r-kernel and regular open (respectively, regular closed) sets.*

3 Almost Contra-1-Somewhat Continuous Functions

Definition 3.1 *A function $f : X \to Y$ is said to be almost contra-1-somewhat continuous provided that for every regular closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists an open set $U \subseteq X$ such that $\emptyset \neq U \subseteq f^{-1}(F)$.*

Theorem 3.2 *If $f : X \to X$ has the property that $f(F) \subseteq F$ for every closed set $F \subseteq X$, then f is almost contra-1-somewhat continuous.*

Proof. Let $F \subseteq X$ be regular closed and assume that $f^{-1}(F) \neq \emptyset$. Since $F = \text{Cl}(\text{Int}(F)$, $\text{Int}(F) \neq \emptyset$. Then, since $\text{Int}(F) \subseteq f^{-1}(F)$, f is contra-1-somewhat continuous.

Corollary 3.3 *The identity mapping $f : X \to X$ is almost contra-1-somewhat continuous.*

Example 3.4 *Let X denote the real numbers with the usual topology. It follows from Corollary 3.3 that the identity mapping $f : X \to X$ is almost contra-1-somewhat continuous. Since $Int(f^{-1}(\{0\})) = \emptyset$, f is not contra-1-somewhat continuous. Since the usual topology coincides with the δ-topology, it is also true that $f : X \to X_\delta$ is not contra-1-somewhat continuous.*

Thus almost contra-1-somewhat continuity does not imply contra-1-somewhat continuity. Nor does it imply contra-1-somewhat continuity when the topology on the codomain is replaced with the δ-topology. However, obviously, if $f : X \to Y_\delta$ is contra-1-somewhat continuous, then $f : X \to Y$ is almost contra-1-somewhat continuous.

Theorem 3.5 *For a function $f : X \to Y$ the following conditions are equivalent:*

(a) *f is almost contra-1-somewhat continuous.*

(b) *For every regular open set $V \subseteq Y$ such that $f^{-1}(V) \neq X$, there exists a closed set $F \subseteq X$ such that $F \neq X$ and $f^{-1}(V) \subseteq F$.*

(c) *For every dense set $M \subseteq X$, r-ker($f(M)$) = r-ker($f(X)$).*

Proof. (a) $\Rightarrow$ (b). Let $V \subseteq Y$ be regular open such that $f^{-1}(V) \neq X$. Then, since $f^{-1}(Y - V) \neq \emptyset$ and $Y - V$ is regular closed, there exists an open set $U \subseteq X$ such that $\emptyset \neq U \subseteq f^{-1}(Y - V)$. Thus $f^{-1}(V) \subseteq X - U$. Since $U \neq \emptyset$, $X - U \neq X$ an thus $X - U$ is the desired set.

(b) $\Rightarrow$ (c). Let M be a dense subset of X and let $V \subseteq X$ be regular open, Assume $f(X) \nsubseteq V$. Then $X \nsubseteq f^{-1}(V)$ and by (b) then exists a closed set $F \subseteq X$ such that $F \neq X$ and $f^{-1}(V) \subseteq F$. Since M is dense in X, $M \nsubseteq F$ and hence $M \nsubseteq f^{-1}(V)$ and $f(M) \nsubseteq V$. Thus r-ker($f(X)$) $\subseteq$ r-ker($f(M)$) and hence r-ker($f(X)$) = r-ker($f(M)$).

(c) $\Rightarrow$ (a). Assume f is not almost contra-1-somewhat continuous. Then there exists a regular closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, and for every open set $U \subseteq X$ such that $U \neq \emptyset$, $U \nsubseteq f^{-1}(F)$. Then for every nonempty open set $U \subseteq X$, $U \cap (X - f^{-1}(F)) \neq \emptyset$. Thus $X - f^{-1}(F)$ is dense in X. Therefore using (c) we have $f(X) \subseteq$ r-ker($f(X)$) = r-ker($f(X - f^{-1}(F))$) $\subseteq$ r-ker($Y - F$) = $Y - F$ and thus $f(X) \subseteq Y - F$ which implies that $f^{-1}(F) = \emptyset$ which is a contradiction. Thus f is almost contra-1-somewhat continuous.

Corollary 3.6 *If $f : X \to Y$ is almost contra-1-somewhat continuous and surjective, then r-ker($f(M)$) = Y for every dense subset M of X.*

Corollary 3.7 *If $f : X \to Y$ is almost contra-1-somewhat continuous and $f(X)$ is regular open in Y, then r-ker($f(M)$) = $f(X)$ for every dense subset M of X.*

Theorem 3.8 *If $f : X \to Y$ is almost contra semicontinuous, then f is almost contra-1-somewhat continuous.*

Proof. Let $F \subseteq Y$ such that F is regular closed and $f^{-1}(F) \neq \emptyset$. Since $f^{-1}(F)$ is semiopen, $f^{-1}(F) \subseteq \text{Cl}(\text{Int}(f^{-1}(F)))$ and thus $\text{Int}(f^{-1}(F)) \neq \emptyset$. Hence f is almost contra-1- somewhat continuous.

Corollary 3.9 *If $f : X \to Y$ is contra almost continuous, then f is almost contra-1-somewhat continuous.*

Corollary 3.10 *If $f : X \to Y$ is contra continuous, then f is almost contra-1-somewhat continuous.*

Corollary 3.11 *If $f : X \to Y$ is an R-map, then f is almost contra-1-somewhat continuous.*

Theorem 3.12 *If $f : X \to Y$ is almost semicontinuous and X has no proper open dense set, then f is almost contra-1-somewhat continuous.*

Proof. Let $V \subseteq Y$ be regular open such that $f^{-1}(V) \neq X$. Then since $f^{-1}(V)$ is semiopen, $f^{-1}(V) \subseteq \text{Cl}(\text{Int}(f^{-1}(V)))$. Since $\text{Int}(f^{-1}(V))$ is open, it is not dense in X. Then $f^{-1}(V) \subseteq \text{Cl}(\text{Int}(f^{-1}(V))) \neq X$ and therefore by Theorem 3.5(b) f is almost contra-1-somewhat continuous.

Theorem 3.13 *If $f : X \to Y$ is almost contra-1-somewhat continuous and A is a dense subset of X, then $f|_A : A \to Y$ is almost contra-1-somewhat continuous.*

Proof. Let $F \subseteq Y$ be regular closed such that $f|_A^{-1}(V) \neq \emptyset$. Then $f^{-1}(V) \neq \emptyset$. Since f is almost contra-1-somewhat continuous, there exists an open set $U \subseteq X$ such that $\emptyset \neq U \subseteq f^{-1}(F)$. Then $U \cap A \subseteq f|_A^{-1}(F)$ and, since A is dense in X, $U \cap A \neq \emptyset$. Therefore $f|_A : A \to Y$ is almost contra-1-somewhat-continuous.

Theorem 3.14 *If A is a open dense subset of X and $f : A \to Y$ is almost contra-1-somewhat-continuous and r-ker$(f(A)) = Y$ then any extension $g : X \to Y$ of f is almost contra-1-somewhat-continuous.*

Proof. Assume $g : X \to Y$ is an extension of f. Let F be a regular closed subset of Y such that $g|_A^{-1}(F) \neq \emptyset$. Since r-ker$(f(A)) = Y$, $f(A) \not\subseteq Y - F$. Then $f(A) \cap F \neq \emptyset$ and hence $f^{-1}(F) \neq \emptyset$. Then there exists a set $U \subseteq A$ such that U is open in A and $\emptyset \neq U \subseteq f^{-1}(F) \subseteq g^{-1}(F)$. Since A is open in X, U is open in X. Therefore $g : X \to Y$ is almost contra-1-somewhat-continuous.

The proof of the following result is straightforward.

Theorem 3.15 *If A and B are open subsets of X such that $X = A \cup B$ and $f : X \to Y$ is a function with the property that both $f|_A : A \to Y$ and $g|_B : B \to Y$ are almost contra-1-somewhat-continuous, then f is almost contra-1-somewhat-continuous.*

4 Almost Contra-2-Somewhat Continuous Functions

Definition 4.1 *A function $f : X \to Y$ is said to be almost contra-2-somewhat continuous if for every regular open set $V \subseteq Y$ such that $f^{-1}(V) \neq \emptyset$, there exists a closed set $F \subseteq X$ such that $\emptyset \neq F \subseteq f^{-1}(V)$.*

Theorem 4.2 *A function $f : X \to Y$ is almost contra-2-somewhat continuous if and only if $f : X \to Y_\delta$ is contra-2-somewhat continuous.*

Proof. To prove the sufficiency assume that $f : X \to Y$ is almost contra-2-somewhat continuous. Let $V \subseteq Y$ be δ-open such that $f^{-1}(V) \neq \emptyset$. Then there exists a regular open subset W of Y such that $W \subseteq V$ and $f^{-1}(W) \neq \emptyset$. Since $f : X \to Y$ is almost contra-2-somewhat continuous, there exists a closed set $F \subseteq X$ such that $\emptyset \neq F \subseteq f^{-1}(W) \subseteq f^{-1}(V)$. Thus $f : X \to Y_\delta$ is contra-2-somewhat continuous.

The proof of the necessity is straightforward.

Obviously contra-2-somewhat continuity implies almost contra-2-somewhat continuity. The following examples show that the converse does not hold and that contra-2-somewhat continuity and contra-1-somewhat continuity are independent of each other.

Example 4.3 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$. Obviously he identity mapping $f : X \to X$ is almost contra-2-somewhat continuous. However, since $f^{-1}(\{a\})$ does not contain a nonempty closed set, f is not contra-2-somewhat continuous.*

Example 4.4 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. The identity mapping $f : X \to X$ is almost contra-1-somewhat continuous by Corollary 3.3, but, since $f^{-1}(\{a\})$ does not contain a nonempty closed set, f is not contra-2-somewhat continuous.*

Example 4.5 *Let X denote the real numbers with the usual topology. Since X is T_1, the function $f : X \to X$ given by $f(x) = 4$ if $x \neq 0$ and $f(0) = 2$ is contra-2-somewhat continuous. However, since $f^{-1}([1, 3])$ does not contain a nonempty open set, f is not contra-1-somewhat continuous.*

Definition 4.6 *Let $A \subseteq B \subseteq X$. Then A is said to be δ-dense in B provided that for every δ-open set $U \subseteq X$, if $U \cap B \neq \emptyset$, then $U \cap A \neq \emptyset$. (That is, that A is dense in B with respect to the relative topology induced on B by the δ-topology on X.)*

The following theorem is a consequence of Theorem 4.2 and Theorem 4.1 [1].

Theorem 4.7 *For a function $f : X \to Y$ the following conditions are equivalent:*

(a) *f is almost contra-2-somewhat continuous.*

(b) *For every δ-closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists an open set $U \subseteq X$ such that $U \neq X$ and $f^{-1}(F) \subseteq U$.*

(c) *For every regular-closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists an open set $U \subseteq X$ such that $U \neq X$ and $f^{-1}(F) \subseteq U$.*

(d) *For every $A \subset X$, if the ker$(A) = X$, then $f(A)$ is δ-dense in $f(X)$.*

Theorem 4.8 *If $f : X \to Y$ is contra almost continuous, then f is almost contra-2-somewhat continuous.*

The proof is straightforward.

Definition 4.9 *A function $f : X \to Y$ is said to be almost somewhat continuous, if for every regular open subset V of Y such that $f^{-1}(V) \neq \emptyset$, there exists a nonempty open subset U of X such that $U \subseteq f^{-1}(V)$.*

Theorem 4.10 *If $f : X \to Y$ is almost somewhat continuous and almost contra-precontinuous, then f is almost contra-2-somewhat continuous.*

Proof. Let V be a regular open subset of Y such that $f^{-1}(V) \neq \emptyset$. Since f is almost somewhat continuous, $\mathrm{Int}(f^{-1}(V)) \neq \emptyset$. Since f is almost contra-precontinuous, $f^{-1}(V)$ is preclosed and hence $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subseteq f^{-1}(V)$. Since $\mathrm{Int}(f^{-1}(V)) \neq \emptyset$, $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \neq \emptyset$. Hence f is almost contra-2-somewhat continuous.

Definition 4.11 *A space X is said to be quasiregular* [14] *if for every nonempty open set $V \subseteq X$, there exists a nonempty open set U such that Cl$(U) \subseteq V$.*

Theorem 4.12 *If X is quasiregular and $f : X \to Y$ is almost somewhat continuous, then f is almost contra-2-somewhat continuous.*

Proof. Let V be a regular open subset of Y such that $f^{-1}(V) \neq \emptyset$. Since f is almost somewhat continuous. $\mathrm{Int}(f^{-1}(V)) \neq \emptyset$. Because X is quasiregular, there exists a nonempty open set U such that $\mathrm{Cl}(U) \subseteq \mathrm{Int}(f^{-1}(V))$. Thus we have $\emptyset \neq \mathrm{Cl}(U) \subseteq f^{-1}(V)$, which proves that f is almost contra-2-somewhat continuous.

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