**International Journal of Contemporary Mathematical Sciences**
**Vol. 13, 2018, no. 3, 125 - 131**
**HIKARI Ltd, www.m-hikari.com**
**https://doi.org/10.12988/ijcms.2018.8412**

# Some Range-Kernel Orthogonality Results for Generalized Derivation

**Ahmed Bachir and Nadiah Zafer Al-Shehri**

Department of Mathematics
King Khalid University, Abha, Saudi Arabia



**Abstract**

In this paper, some range-kernel orthogonality results to $p$-$w$-hyponormal operators and $(\mathcal{Y})$ or dominant operators are given, also we will generalize some commutativity results.

**Mathematics Subject Classification:** 47B47, 47A30, 47B20

**Keywords:** Fuglede-Putnam Theorem, $p$-$w$-hyponormal, Derivation, Orthogonality, Putnam-Fuglede's Theorem

## 1. Introduction

For complex spaces $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the algebra of all bounded operators on $\mathcal{H}$ , the algebra of all bounded operators on $\mathcal{K}$ and the set of all bounded transformations from $\mathcal{H}$ to $\mathcal{K}$ respectively.

A bounded operator $A \in \mathcal{B}(\mathcal{H})$, set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A^*|^2 - |A|^2$ ( the self commutator of $A$) and consider the following definitions: $A$ is normal if $A^*A = AA^*$, hyponormal if $A^*A \geq AA^*$, $p$-hyponormal if $|A|^{2p} \geq |A^*|^{2p}$ for $0 < p < 1$ and semi-hyponormal if $|A| \geq |A^*|$. The lowner-Heinz inequality implies that if $A$ is $q$-hyponormal, then $A$ is $p$-hyponormal for all $0 < p \leq q$. An invertible operator $A \in \mathcal{B}(\mathcal{H})$ is called *log*-hyponormal if $\log(A^*A) = \log(AA^*)$. Clearly every invertible $p$-hyponormal operator is log-hyponormal.

Let $A = U|A|$ be the polar decomposition of $A$. A. Aluthge[1] defined the operator $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ which is called the Aluthge transformation of $A$. An operator $A$ is said to be $w$-hyponormal if $|\widetilde{A}| \geq |A| \geq |\widetilde{A}^*|$ . An operator $A$ is said to be $p$-$w$-hyponormal $(0 < p \leq 1)$ if $|\widetilde{A}|^p \geq |A|^p \geq |\widetilde{A}^*|^p$ [13, 16]. It is well known that the class of $w$-hyponormal operators contains, both $p$-and log-hyponormal operators [2]. These classes are related by proper inclusion

hyponormal $\subset$ $p$-hyponormal $\subset$ $w$-hyponormal $\subset$ $p$-$w$-hyponormal.

It is well known that if $A$ is $w$-hyponormal, then $\widetilde{A}$ is semi- hyponoormal and if $A$ is $p$-$w$-hyponormal, then $\widetilde{A}$ is $\frac{p}{2}$- hyponormal [16].

An operator $A$ is said to be class $\mathcal{Y}_\alpha$ for $\alpha \geq 1$ if there exist a positive number $k_\alpha$ such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda)^*(A - \lambda) \text{ for all } \lambda \in \mathbb{C}.$$

If $1 \leq \alpha \leq \beta$, then $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be dominant if for each $\lambda \in \mathbb{C}$ there exists a positive number $M_\lambda$ such that

$$(A - \lambda)(A - \lambda)^* \leq M_\lambda (A - \lambda)^*(A - \lambda).$$

If the constants $M_\lambda$ are bounded by a positive operator $M$, then $A$ is said to be $M$-hyponormal. Evidently $M$-hyponormal operators are dominant. Let $\mathcal{Y} = \bigcup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that $M$-hyponormal are class $\mathcal{Y}_2$ [15].

The famous Fuglede-Putnam theorem [8] asserts that if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are normal and $AC = BC$ for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H}$, then $A^*C = CB^*$. Fuglede-Putnam's Theorem for $p$-hyponormal operators and $\mathcal{Y}$ operators was proven by Mecheri et. al [11], and extensively studied in [4] for $w$-hyponormal operators and $\mathcal{Y}$.

Let $A, B \in \mathcal{B}(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$ induced by $A$ and $B$ as follows

$$\delta_{A,B}(X) = AX - XB \text{ for all } X \in \mathcal{B}(\mathcal{H}) \tag{1}$$

J. Anderson and C. Foias [3] proved that if $A$ and $B$ are normal, $S$ is an operator such that $AS = SB$, then

$$||\delta_{A,B}(X) - S|| \geq ||S|| \text{ for all } X \in \mathcal{B}(\mathcal{H})$$

where $\|.\|$ is the usual operator norm. Hence the range of $\delta_{AB}$ is orthogonal to the null space of $\delta_{AB}$. The orthogonality here is understood to be in the sense of Birkhoff-James [3].

The purpose of this paper is to investigate the orthogonality of ran$(\delta_{A,B})$ and ker$(\delta_{A,B})$ for certain operators. We prove that ran$(\delta_{A,B})$ is orthogonal to ker$(\delta_{A,B})$ when

(i) If $A$ is dominant and $B^*$ is an injective $p$-$w$-hyponormal;
(ii) $A$ is an injective $p$-$w$-hyponormal and $B^*$ is a dominant operator;
(iii) $A$ is an injective $p$-$w$-hyponormal and $B^*$ is $\mathcal{Y}$ class operator;
(iv) $A$ and $B^*$ are $*$-paranormal operators.

## 2. Preliminaries

The following lemmas and Theorems are useful for the sequel.

**Lemma 1.** [16] *Let $A$ be p-w-hyponormal operator. If $\widetilde{A}$ is normal, then $A = \widetilde{A}$.*

**Lemma 2.** [13] *Let $A$ be p-w-hyponormal operator and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace of $A$. Then the restriction $A \mid \mathcal{M}$ is p-w-hyponormal.*

**Theorem 3.** [?] *Let $A \in \mathcal{B}(\mathcal{H})$ be dominant and $B^* \in B(K)$ be p-hyponormal or* log*-hyponormal. If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $A^*C = CB^*$.*

**Theorem 4.** [13] *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective p-w-hyponormal ($0 < p \leq 1$) and $B^* \in \mathcal{B}(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$.*

**Theorem 5.** [13] *Let $A \in \mathcal{B}(\mathcal{H})$ be p-w-hyponormal ($0 < p \leq 1$) such that $\ker A \subset \ker A^*$ and $B^* \in \mathcal{B}(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$.*

**Theorem 6.** [12] *Let $A \in \mathcal{B}(\mathcal{H})$ be $*$-paranormal and $B^* \in \mathcal{B}(\mathcal{K})$ be $*$-paranormal. If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$.*

## 3. Main Results

**Definition 7.** We say that $A \in B(H)$ is finite if the distance $\operatorname{dist}(I, \operatorname{ran}(\delta_A)) \geq 1$ from the identity $I$ to the range of $\delta_A$.

**Definition 8.** Let $A \in \mathcal{B}(\mathcal{H})$, the reduisant approximate point spectrum denoted $\sigma_{ra}(A)$ is the set of scalars $\lambda \in \mathbb{C}$ for which there exists a normalized sequence $\{x_n\} \subset \mathcal{H}$ verifying

$$(A - \lambda)x_n \to 0 \text{ and } (A - \lambda)^* x_n \to 0$$

**Remark 9.** The reduisant approximate point spectrum $\sigma_{ra}(A)$ coincides with the approximate point spectrum $\sigma_a(A)$, when $A$ is

(i) $*$-paranormal [12].
(ii) dominant [5].
(iii) $p$-$w$-hyponormal [16].

**Proposition 10.** *Let $A \in \mathcal{B}(\mathcal{H})$. If $\sigma_{ra}(A)$ is not empty, then $A$ is finite.*

*Proof.* Let $\lambda \in \sigma_{ra}(A)$ and $\{x_n\}$ be a normalized sequence in $\mathcal{H}$ such that

$$(A - \lambda)x_n \to 0 \text{ and } (A - \lambda)^* x_n \to 0.$$

If $X \in \mathcal{B}(\mathcal{H})$, then we have

$$\begin{aligned} \|AX - XA - I\| &= \|(A - \lambda)X - X(A - \lambda) - I\| \\ &\geq |\langle (A - \lambda)Xx_n, x_n\rangle - \langle X(A - \lambda)x_n, x_n\rangle - 1|. \end{aligned}$$

Letting $n \to \infty$, yields $\|AX - XA - I\| \geq 1$. □

**Corollary 11.** *Every $*$-paranormal (resp. dominant or p-w-hyponormal) operator is finite.*

*Proof.* See [12](resp. [5] or [15, 16]). □

Now, we give the range-kernel orthogonality for certain operator classes.

**Proposition 12.** *Let $A \in \mathcal{B}(\mathcal{H})$ and $N$ is a normal operator such that $NA = NA$, then for every $\lambda \in \sigma_p(N)$ set of eigenvalues)*

$$|\lambda| \leq \operatorname{dist}(N, \operatorname{ran}(\delta_A)).$$

*holds when $A$ is*

(i) *dominant;*
(ii) *p-w-hyponormal*
(iii) *$*$-paranormal;*

*Proof.* Let $\lambda \in \sigma_p(N)$ and $M_\lambda$ be the eigenspace associated to $\lambda$. Since $NA = AN$, then $N^*A = AN^*$ by Fuglede's Theorem [8]. Hence $M_\lambda$ reduces $N$ and $A$ by

(i) [12];
(ii) [5];
(iii) [15].

Let $T \in \mathcal{B}(\mathcal{H})$, according to the decomposition of $\mathcal{H} = M_\lambda \oplus M_\lambda^\perp$, we can write $A$, $N$ and $T$ as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

Hence

$$\begin{aligned} \|N + AT - TA\| &= \left\| \begin{bmatrix} \lambda + A_1T_1 - T_1A_1 & * \\ * & * \end{bmatrix} \right\| \\ &\geq \|\lambda + A_1T_1 - T_1A_1\| \\ &\geq |\lambda| \left\| I + A_1\left(\frac{T_1}{\lambda}\right) - \left(\frac{T_1}{\lambda}\right)A_1 \right\| \\ &\geq |\lambda|. \end{aligned}$$

□

In what follows, we need the Berberian technique, it allows us to construct a Hilbert space which contains a given Hilbert space $H$ on which we could speak about "approached eigenvectors" and those as eigenvectors.

**Theorem 13.** [6] *Let $\mathcal{H}$ be a complex Hilbert space, then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ $(T \mapsto T^0)$ satisfying: $\varphi$ is an $^*$-isometric isomorphism preserving order. Moreover*

(1) $\sigma(T) = \sigma(T^0)$
(2) $\sigma_a(T) = \sigma_a(T^0) = \sigma_p(T^0)$.

**Lemma 14.** *If $T \in \mathcal{B}(\mathcal{H})$ is $*$-paranormal (resp. dominant or p-w-paranormal), then $T^0 \in B(K)$ is $*$-paranormal (resp. dominant or p-w-hyponormal).*

*Proof.* The proof is easy by applying the definition of $*$-paranormal (resp. dominant or $p$-$w$-hyponormal) operator. □

**Theorem 15.** *If $A$ is dominant (resp. p-w-hyponormal or $*$-paranormal ), then for every normal operator $N$ such that $AN = NA$, we have*

$$\|N\| \leq \operatorname{dist}(N, \operatorname{ran}(\delta_A)).$$

*Proof.* Let $\lambda \in \sigma_p(A) = \sigma_a(A)$ [5], then from Proposition 14, $N^0$ is normal, $A^0$ (resp. $*$-paranormal or $p$-$w$-hyponormal), and $A^0N^0 = N^0A^0$, also $\lambda \in \sigma_p((A^0)$. By Proposition 12, we get for every $T \in B(H)$

$$|\lambda| \leq \|N^0 + A^0T^0 - T^0A^0\|$$

implying

$$\sup_{\lambda\in\sigma(N^0)} |\lambda| = \|N^0\| = \|N\| = r(N) \leq \|N + AT - TA\|. \tag{2}$$

□

**Theorem 16.** *Let $A \in B(H)$ and $B \in \mathcal{B}(\mathcal{H})$. If one of the following assertions*

(i) *$A$ is dominant and $B^*$ is p-hyponormal or* log*-hyponormal;*
(ii) *$A$ is an injective p-w-hyponormal and $B^*$ is $\mathcal{Y}$ class operator;*
(iii) *$A$ is p-w-hyponormal ($0 < p \leq 1$) such that* $\ker A \subset \ker A^*$ *and $B^*$ is class $\mathcal{Y}$ operator;*
(iv) *$A$ and $B^*$ are $*$-paranormal operators;*

*is verified, then for every $T \in \ker(\delta_{A,B})$, we have $\|T\| \leq \operatorname{dist}(T, \operatorname{ran}(\delta_{A,B}))$.*

*Proof.* The pair $(A, B)$ verify the Fuglede-Putnam Theorem's for (i) (resp. (ii)) from Theorem 3 (resp. Theorem 4), for (iii) from Theorem 5 and (iv) from Theorem 6, equivalently, if $T \in \ker(\delta_{A,B}$, then $T \in \ker(\delta_{A^*,B^*}$. Thus,

$$ATT^* = TBT^* = TT^*A.$$

Applying Theorem 15, we obtain for all $T \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \|TT^*\| &= \|T\|^2 \\ &\leq \|TT^* + AXT^* - XT^*A\| \\ &\leq \|TT^* + AXT^* - XBT^*\| \\ &\leq \|T^*\|\|T + AX - XB\| \end{aligned}$$

Hence

$$\|T\| \leq \|T + AX - XB\|.$$

□

Next, we prove some commutativity results. A.H. Moadjil [10] proved that if $N$ is normal operator such that $N^2X = XN^2$ and $N^3X = XN^3$, for some $X \in B(H)$, then $NX = XN$. In [10], A.H. Moadjil gave a counterexample for proving that this result is not true for quasinormal operators $(A(A^*A) = (A^*A)A$. F. Kittaned [9] extend this result for subnormal operators by taking $A$ and $B^*$ subnormal operators, i.e., if $A^2X = XB^2$ and $A^2X = XB^2$ for some $X \in B(H)$, then $AX = XB$.

This result can be generalized to some several operator classes as follows:

**Theorem 17.** *Let $A, B \in B(H)$. Assume that*

(i) *If $A$ is dominant and $B^*$ is p-hyponormal or* log*-hyponormal, or*
(ii) *$A$ is an injective p-w-hyponormal and $B^*$ is a dominant operator, or*
(iii) *$A$ is p-w-hyponormal ($0 < p \leq 1$) such that* $\ker A \subset \ker A^*$ *and $B^*$ is class $\mathcal{Y}$, or*
(iv) *$A$ and $B^*$ are $*$-paranormal operators.*

*If $A^2X = XB^2$ and $A^2X = XB^2$ for some $X \in B(H)$, then $AX = XB$.*

*Proof.* Let $T = AX - XB$, then

$$\begin{aligned} A^2T &= A^3X - A^2XB) \quad (3) \\ &= XB^3 - XB^3 \\ &= 0 \\ TB^2 &= AXB^2 - XB^3 \quad (4) \\ &= A^3X - A^3X \\ &= 0, \end{aligned}$$

and

$$ATB = A^2XB - AXB^2 = XB^2 - A^2X = 0. \quad (5)$$

Hence, from (3), (4) and (5) we get

$$A(AT - TB) = A^2T - ATB = 0 \quad (6)$$

and

$$(AT - TB)T = ATB - TB^2 \quad (7)$$

This yields $AT - TB \in \ker(\delta_{A,B}) \cap \operatorname{ran}(\delta_{A,B}) = \{0\}$, therefore $AT - TB = 0$. Hence by Theorem (16) $T \in \ker(\delta_{A,B}) \cap \operatorname{ran}(\delta_{A,B}) = \{0\}$, i.e., $T = 0$ and so $AX = XB$ as desired. □

## 4. Open Problem

The open problem here is to find classes of nonnormal of operators satisfying the Fuglede-Putnam Property and consequently to obtain the range kernel orthogonality and commutativity results.

## References

[1] A. Aluthge, On $p$-hyponormal operators for $0 < p < 1$, *Integral Equations Operator Theory*, **13** (1990), 307 - 315. https://doi.org/10.1007/bf01199886

[2] A. Aluthge and D. Wang, On $w$-hyponormal operators, *Integral Equations Operator Theory*, **36** (2000), 1 -10. https://doi.org/10.1007/bf01236285

[3] J. Anderson and C. Foias, Properties which normal normal operators share with normal derivations and related operators, *Pacific J. Math.*, **61** (1975), 313325. https://doi.org/10.2140/pjm.1975.61.313

[4] A. Bachir, Fuglede-Putnam's theorem for $p$-hyponormal or $\mathcal{Y}$ operators, *Ann. Funct. Anal.*, **4** (2013), no. 1, 53-60. https://doi.org/10.15352/afa/1399899836

[5] A. Bachir, Dérivations généralisées induites par certaines classes d'opérateurs, *Maghreb Math. Rev.*, **189** (1999), no. 8, 11-17.

[6] S.K. Berberian, Approximate proper vectors, *Proc. Amer. Math. Soc.*, **13** (1962), 111–114. https://doi.org/10.1090/s0002-9939-1962-0133690-8

[7] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413 - 415. https://doi.org/10.1090/s0002-9939-1966-0203464-1

[8] B. Fuglede, A commutativity theorem for normal operators, *Proc. Nat. Acad. Sci.*, **36** (1950), 35-40. https://doi.org/10.1073/pnas.36.1.35

[9] B. F. Kittaneh, On generalized Fuglede-Putnam Theorem of Hilbert-Schmidt type, *Proc. Amer. Math. Soc.*, **88** (1983), 293-298. https://doi.org/10.1090/s0002-9939-1983-0695261-8

[10] A.L. Moadjil, On the commutant of relatively prime power in Banach algebra, *Proc. Amer. Math. Soc.*, **57** (1976), 243-249. https://doi.org/10.1090/s0002-9939-1976-0407612-6

[11] S. Mecheri, K. Tanahashi and A. Uchiyama, Fuglede - Putnam's theorem for p - hyponormal or class $\mathcal{Y}$ operators, *Bull. Korean. Math. Soc.*, **43** (2006), 747-753. https://doi.org/10.4134/bkms.2006.43.4.747

[12] P. Pagacz, The Putnam-Fuglede property for paranormal and *-paranormal operators, *Opuscula Math.*, **33** (2013), no. 3, 565–574. https://doi.org/10.7494/opmath.2013.33.3.565

[13] T. Prasad and A. Bachir, Fuglede-Putnam theorem for $p$-$w$-hyponormal or class $\mathcal{Y}$ operators, *IJOPCM*, **11** (2018), 17-24.

[14] J. G. Stampfli and B. L. Wadhwa, On dominant operators, *Monatsh. Math.*, **84** (1977), 143 - 153. https://doi.org/10.1007/bf01579599

[15] Yang Changsen, Li Haiying, Properties of $p$-$w$-hyponormal operators, *Appl. Math. J. Chinese Univ.*, **21** (2006), no. 1, 64-68. https://doi.org/10.1007/s11766-996-0024-1

[16] Yang Changsen and Li Haiying, A note on $p$-$w$-hyponormal operators, *Acta Mathematica Sinica (Chinese)*, **49** (2006), no. 1, 19-25.

**Received: May 1, 2018; Published: May 18, 2018**