On Annihilator Small Intersection Graph

Mehdi Sadiq Abbas and Hiba Ali Salman
Department of Mathematics
College of Science, Al-Mustansiriyah University
Baghdad, Iraq

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Abstract

Consider $R$ as an associative ring with non-zero identity and $M$ be a left $R$-module. We define the annihilator small intersection graph $G_{a}(M)$ of $M$ with all non annihilator small proper submodules of $M$ as vertices and two distinct vertices $N,K$ are adjacent if $N \cap K$ is non annihilator small submodule of $M$. In this paper we investigate the effect of the algebraic properties of $M$ on the graph-theoretic properties.

Keywords: a-small intersection graph, Null graph, Empty graph, The Girth of a graph

1. Introduction

Throughout this paper, $R$ will denote an associative ring with identity, and $M$ a left $R$-module. Remember that, a submodule $S$ of an $R$-module $M$ is said to be small if for each submodule $T$ of $M$ with $S+T=M$ then $T=M$, and the Jacobson radical of an $R$-module $M$ denoted by $J(M)$ is defined as the sum of all small submodules of $M$, or as the intersection of all maximal submodules of $M$ [8]. Also, $M$ is called hollow if every proper submodule of it is small [8]. The authors in [1] introduced and studied a generalization of small submodules, where a submodule $N$ of an $R$-module $M$ is called annihilator small (briefly a-small) in $M$ (denoted by $N a\ll M$), in case for every submodule $L$ of $M$, $N+L=M$ implies that $\text{ann}(L)=\text{ann}(M)$. They also defined $J_{a}(M)$ as the sum of all a-small submodules of $M$. The set of all maximal submodules of an $R$-module $M$ is denoted by $\text{Max}(M)$. We shall classify $\text{Max}(M)$ in our work into two parts, the first contains the maximal submodules that $J_{a}(M)$ is contained in each one of them (denoted by $\mu(J_{a})$), the second contains the maximal submodules that don’t contain $J_{a}(M)$ in any of them.
A graph $G_a$ is defined as the pair $(V(G_a), E(G_a))$, where $V(G_a)$ is the set of vertices of $G_a$ and $E(G_a)$ is the set of edges of $G_a$. The following definitions are mentioned in [3] and we will state them with a slight change for the sake of completion of the work. For any two distinct vertices $a$ and $b$ then $a - b$ means that $a$ and $b$ are adjacent. The degree of a vertex $a$ of a graph $G_a$ (denoted by $\text{deg}(a)$) is the number of edges incident on $a$. If $|V(G_a)| \geq 2$, a path from $a$ to $b$ is a series of adjacent vertices $a - v_1 - v_2 - ... - v_n - b$. The path that starts with a vertex $a$ and ends with it is called a cycle. Note that a graph whose vertices set is empty is called a null graph and a graph whose edge-set is empty is called an empty graph. A graph $G_a$ is called connected if for any vertices $a$ and $b$ of $G_a$ there is a path between $a$ and $b$, otherwise $G_a$ is called disconnected. The girth of $G_a$, is the length of the shortest cycle in $G_a$ and it is denoted by $g(G_a)$. If $G_a$ has no cycle, we define the girth of $G_a$ to be infinite. A graph $G_a$ is called complete if every pair of distinct vertices are adjacent.

We have introduced in [1] the concept of a-small submodules and studied some of their properties, as well as their relation with other types of small submodules, where it is clear from the definitions of both small and a-small submodules that every small submodule is a-small, but the converse is not true generally [1] contains examples for that. In 2012, the authors in [2] have established the intersection graph of submodules of a module and studied its properties, their aim was to study the connection between the algebraic properties of a module and the graph theoretic properties associated to it. In 2016, the small intersection graph relative to multiplication modules has been studied by the authors in [3]. This has motivated us to present this work, where we investigate some of the results appeared in [3] with our definition of a-small submodules presented earlier. But our work is not relative to multiplication modules as that in [3]. In this work, we shall study the effect of the algebraic properties of $M$ on the graph theoretic properties of $G_a(M)$.

## 2. Results of $G_a(M)$

**Definition 2.1:** let $M$ be an $R$-module. We define the a-small intersection graph of $M$ denoted by $G_a(M)$ with all proper non a-small submodules as vertices, and two distinct vertices $N$ and $K$ are adjacent if $N \cap K$ is non-a-small in $M$.

The following two lemmas appears in [1]

**Lemma 2.2:** Let $M$ be an $R$-module, with submodules $A \subseteq N$. If $N \ll M$, then $A \ll M$.

**Lemma 2.3:** Let $M$ be an $R$-module. Then $J_a(M)$ is a submodule of $M$ that contains every a-small submodule of $M$.

Recall that a proper submodule $N$ of an $R$-module $M$ is called prime if $ax \in N$ for $a \in R$ and $x \in M$, then either $aM \subseteq N$ or $x \in N$. In particular, if $N$ is a prime
submodule of $M$, then $P = [N:M]$ is a prime ideal of $R$. Moreover, every maximal submodule of $M$ is a prime submodule [7].

We shall consider the following condition.

**Condition**: If $N$ & $K$ are two a-small submodules of $M$ and $P$ a prime submodule of $M$ with $N \cap K \subseteq P$ then either $N \subseteq P$ or $K \subseteq P$.

It seems important to mention that the above condition is satisfied in multiplication modules [5].

**Proposition 2.4**: Let $M$ be an $R$-module, which satisfies condition*, and $\text{Max}(M) = \{M_i\}_{i \in I}$ where $|I| > 1$. If $J$ is a finite subset of $I$ such that $M_j \notin \mu(J_a)$ for each $j \in J$. Then $\cap_{j \in J} M_j$ is non a-small in $M$.

**Proof**: Assume that $\cap_{j \in J} M_j$ is a-small in $M$. Then by lemma (2.3) we get that $\cap_{j \in J} M_j \subseteq J_a(M)$, but $J_a(M) \subseteq \cap \{W | W \in \mu(J_a)\}$, hence $\cap_{j \in J} M_j \subseteq W$, for each $W$ in $\mu(J_a)$. By condition* $M_j \subseteq W$ for some $j \in J$ which contradicts the maximality of $M_j$. Thus $\cap_{j \in J} M_j$ is non a-small in $M$. $\blacksquare$

**Corollary 2.5**: Let $R$ be a commutative ring and $M$ a multiplication $R$-module, and $\text{Max}(M) = \{M_i\}_{i \in I}$ where $|I| > 1$. If $J$ is a finite subset of $I$ such that $M_j \notin \mu(J_a)$ for each $j \in J$. Then $\cap_{j \in J} M_j$ is non a-small in $M$.

Next, we shall state a sufficient condition for $G_a(M)$ being a null graph. But first recall that an $R$-module $M$ is called local if it has a unique maximal submodule, that is, a proper submodule that contains all proper submodules. In this case $J(M)$ equals that largest submodule and $J(M)$ is small in $M$ [8].

It is well known that the Jacobson radical of an $R$-module is the sum of all small submodules of $M$, and characterized also as the intersection of all maximal submodules of $M$. The following lemma that is mentioned in [1] is in this direction.

**Lemma 2.6**: Let $M$ be a finitely generated $R$ –module and $J_a(M)$ a\ll $M$. Then we have the following statements:

1. $J_a(M)$ is the unique largest annihilator small submodule of $M$.
2. $J_a(M) = \cap \{W | W \in \mu(J_a)\}$

**Proposition 2.7**: Let $M$ be a finitely generated $R$-module and $J_a(M)$ a\ll $M$. If $M$ is local, then $G_a(M)$ is a null graph.

**Proof**: Let $W$ be the maximal submodule of $M$. It is clear that $J_a(M) \subseteq W$ and then by lemma (2.6) $J_a(M) = W$. This implies that every proper submodule of $M$ is contained in $J_a(M)$ which is a-small in $M$ by our assumption. Hence, every proper submodule of $M$ is a-small by lemma (2.2), and then $G_a(M)$ is a null graph. $\blacksquare$
The converse of the above proposition may not be true in general. The following examples discuss the above proposition in more than one situation.

Examples 2.8:

a. In the $\mathbb{Z}$-module $\mathbb{Z}$ every proper submodule is a-small [1], this implies that $G_a(\mathbb{Z})$ is a null graph. Although, it is well-known that $\mathbb{Z}$ is not a local module.

b. It is well-known that $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module has no maximal submodules, but $G_a(\mathbb{Z}_{p^\infty})$ is a null graph since every submodule of $\mathbb{Z}_{p^\infty}$ is a-small[1].

Our aim is to study non-null graphs, since all definitions for graph theory are for non-null graphs only.

**Proposition 2.9:** Let $M$ be a finitely generated $R$-module. If $\text{Max}(M)=\{M_1, M_2\}$ where $M_1$ and $M_2$ are both hollow $R$-modules and belong to $\mu(J_a)$, then $G_a(M)$ is an empty graph.

**Proof:** Suppose that $N$ is a non a-small submodule of $M$ with $N \neq M_i$ $i = 1,2$. Since $M$ is finitely generated then $N \subseteq M_1$ or $N \subseteq M_2$, now since both $M_1$ and $M_2$ are hollow modules, we get that $N$ is small in $M$, hence $N \ll M$. Thus $M_1$ and $M_2$ are the only vertices of $G_a(M)$ which are not adjacent, since $M_1 \cap M_2 \ll M_1$ by $M_1$ being hollow, hence $M_1 \cap M_2 \ll M$ and this implies that $M_1 \cap M_2 \ll M$. Hence, $G_a(M)$ is an empty graph. 

The following example shows that the condition that both $M_1$ and $M_2$ are hollow in proposition(2.9)can’t be dropped.

**Example 2.10:** Let $M=\mathbb{Z}_{18}$ as $\mathbb{Z}$-module, then $V(G_a(M))=\{2M, 3M, 9M\}$. Although, $\text{Max}(M)=\{2M, 3M\}$ and both contain $J_a(M)$ which is $6M$, but $3M$ is not hollow since $9M$ is not small in $3M$. clearly, $G_a(M)$ is not an empty graph. See figure (2.1).

![Figure (2.1)](image)

**Proposition 2.11:** Let $M$ be a finitely generated $R$-module that satisfies condition* and $J_a(M) \ll M$. If $\text{Max}(M)=\{M_1, M_2\}$ where both $M_1$ & $M_2 \in \mu(J_a)$, then
\( G_a(M) = G_{a1} \cup G_{a2} \) where \( G_{a1} \) and \( G_{a2} \) are two disjoint complete subgraphs of \( G_a(M) \).

**Proof:** Let \( G_{aj} = \{ L \subseteq M | L \subseteq M_j \ and \ L \) is non a-small in M \} for \( j=1,2 \). Consider \( N, K \in G_{a1} \), we claim that \( N \) and \( K \) are adjacent. Otherwise, if \( N \cap K \ a \ll M \) then \( N \cap K \subseteq J_a(M) = M_1 \cap M_2 \), that is \( N \cap K \subseteq M_1 \) and \( N \cap K \subseteq M_2 \). By the use of condition* we have either \( K \subseteq M_2 \) or \( N \subseteq M_2 \), this implies that either \( K \subseteq M_1 \cap M_2 \) or \( N \subseteq M_1 \cap M_2 \). But \( M_1 \cap M_2 = J_a(M) \) (lemma 2.6). Hence, either \( K \subseteq M \) or \( N \subseteq M \) a contradiction! Thus \( G_{aj}(M) \) \( j=1,2 \)is a complete subgraph.

Next, let \( N \in G_{a1} \) and \( K \in G_{a2} \), assume that \( N \) and \( K \) are adjacent. Since \( J_a(M) \ a \ll M \) and \( N \cap K \subseteq M_1 \cap M_2 = J_a(M) \) we get \( N \cap K \ a \ll M \) (by lemma 2.2) a contradiction! Thus \( G_a(M) = G_{a1} \cup G_{a2} \) where \( G_{a1} \) and \( G_{a2} \) are two complete subgraphs. \( \blacksquare \)

**Corollary 2.12:**

1. Let \( M \) be a finitely generated \( R \)-module that satisfies condition* and \( J_a(M) \ a \ll M \). If \( \text{Max}(M) = \{ M_1, M_2 \} \) where \( M_1 \) and \( M_2 \in \mu(J_a) \) then \( G_a(M) \) is disconnected.

2. Let \( R \) be a commutative ring and \( M \) a multiplication \( R \)-module and \( J_a(M) \ a \ll M \). If \( \text{Max}(M) = \{ M_1, M_2 \} \) where \( M_1 \) and \( M_2 \in \mu(J_a) \) then \( G_a(M) \) is disconnected.

The following example shows that \( |\text{Max}(M)| = 2 \) in the previous proposition can’t be omitted.

**Example 2.13:** Let \( M=\mathbb{Z}_{30} \) as \( \mathbb{Z} \)-module. \( V(G_a(M)) = \{ 2M, 3M, 5M, 6M, 10M, 15M \} \). The figure (2.2) below shows that \( G_a(M) \) is a connected graph while \( \text{Max}(M) = \{ 2M, 3M, 5M \} \) and all contain \( J_a(M) \).

![Figure (2.2)](image)

**Proposition 2.14:** Let \( M \) be a finitely generated \( R \)-module and \( J_a(M) \ a \ll M \). If \( \text{Max}(M) = \{ M_1, M_2 \} \) where both \( M_1 \) and \( M_2 \) belong to \( \mu(J_a) \) and \( G_a(M) \) contains a cycle, then \( g(G_a(M)) = 3 \).
Proof: since Max(M)={M_1,M_2} and both M_1 & M_2 ∈ μ(J_a) ,then by the use of proposition (2.11) we get that G_a(M) is a union of two disjoint complete subgraphs. By using our assumption that G_a(M) contains a cycle we get g(G_a(M))=3. □

Natural questions might be asked are that: What about g(G_a(M)) if |Max(M)| ≥ 3, will it equal 3? What does the property that these maximal submodules belong to μ(J_a) can effect the answer? Answers to these questions are revealed in the following. But first we need the following condition to be stated.

Condition**: For each M_1, M_2 two maximal submodules of M such that M_1 ∈ μ(J_a) and M_2 ∈ μ(J_a), M_1 ∩ M_2 is non-a-small in M.

Lemma 2.15: Let M be a finitely generated R-module with condition**, and Max(M)={M_i} i=1,...,n where n>2. If there exists only one M_j ∈ μ(J_a) j∈ {1,...,ν}, then G_a(M) has no cycle.

Proof: Assume that G_a(M) contains a cycle, that is: there exists M_1,M_2,M_3 ∈ Max(M) such that M_1 − M_2 − M_3 − M_4 is a cycle in G_a(M). Now, this implies that M_1 & M_2 are adjacent, that is: M_1 ∩ M_2 is non-a-small in M, and hence either M_1 = M_j or M_2 = M_j by condition**, or both of them doesn’t contain J_a(M) (proposition 2.4) a contradiction with our assumption that only one maximal submodule of M doesn’t belong to μ(J_a). Without loss of generality we will assume that M_1 = M_j. But we have M_2 − M_3 are also adjacent which implies that M_2 ∩ M_3 is non-a-small in M and in this case both of them doesn’t contain J_a(M) which is again a contradiction. Thus G_a(M) contains no cycle. □

Proposition 2.16: let M be a finitely generated R-module with condition**. If Max(M)={M_i} i=1,...,n where n>2, and at least two components of Max(M) doesn’t belong to μ(J_a), then g(G_a(M))=3.

Proof: Let |Max(M)|≥3 and choose M_1,M_2,M_3 ∈ Max(M), then we have four possibilities. The first is that all of them are containing J_a(M) and in this case no one of them is adjacent to the other. The second is that one of them doesn’t contain J_a(M) and in this case there is no cycle between them by lemma(2.15), the third is that two of them doesn’t contain J_a(M), say M_1 & M_2, then in this case M_1 ∩ M_2 is non-a-small in M by proposition(2.4). Moreover, M_1 ∩ M_3 and M_2 ∩ M_3 are also non-a-small in M by condition**. This implies that M_1 − M_2 − M_3 − M_4 is a cycle in G_a(M) and clearly g(G_a(M)) = 3. The last possibility is that no one of them contains J_a(M) and by the use of proposition(2.4) we get that each two of them are adjacent, which implies again that M_1 − M_2 − M_3 − M_4 is a cycle in G_a(M) and g(G_a(M)) = 3. □

Example (2.13) shows that the assumption that at least two of the maximal submodules don’t contain J_a(M) can’t be omitted, since |Max(M)|=3 in it but each one of them contains J_a(M) and g(G_a(M))=6.
References


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