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A Characterization of the Cactus Graphs with Equal Domination and Connected Domination Numbers

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Abstract

A cactus graph is a connected graph in which any two cycles have at most one vertex in common. Let $\gamma(G)$ and $\gamma_c(G)$ be the domination number and connected domination number of a graph G, respectively. We can see that $\gamma(G) \leq \gamma_c(G)$ for any graph G. S. Arumugam and J. Paulraj Joseph [1] have characterized trees, unicyclic graphs and cubic graphs with equal domination and connected domination numbers. A few years later, Xue-gang Chena, Liang Suna, Hua-ming Xing [3] characterized the cactus graphs for which the domination number is equal to the connected domination number. Their characterization is in terms of global properties of a construction. In this paper, we provide a constructive characterization of the cactus graphs with equal domination and connected domination numbers.

Mathematics Subject Classification: 05C69

Keywords: cactus graph, dominating set, connected dominating set, domination number, connected domination number

1 Introduction

A *dominating set* for a graph G is a subset $S \subseteq V(G)$ such that every vertex not in S is adjacent to at least one member of S (i.e. $N_G[S] = V(G)$). A dominating set S is called a *connected dominating set* if the induced subgraph $\prec S \succ$ is connected. The *domination number* (resp. *connected domination number*) $\gamma(G)$(resp.$\gamma_c(G)$) of G is defined to be the minimum cardinality among all dominating sets (resp. all connected dominating sets) of G. A dominating set of cardinality $\gamma(G)$ in G is said to be a *γ-set*. A connected dominating set of cardinality $\gamma_c(G)$ in G is said to be a *γ_c-set*. A set S is a γ-set and γ_c-set of G, then we call S a *(γ, γ_c)-set of G*.

One of the fastest growing areas within graph theory is the study of domination and related subset problems. A dominating set have been proposed as a virtual backbone for routing in wireless ad hoc networks (see [8]). The topology of such wireless ad hoc network can be modeled as a unit-disk graph (UDG), a geometric graph in which two vertices are adjacent if and only if their distance is at most one. A dominating set of a wireless ad hoc network is a dominating set of the corresponding UDG. The research of domination in graphs are initiated by Ore [7]. Domination and its variations in graphs are well studied, a lot of papers have been written on this topic (see [4],[5],[6]).

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. The (*open*) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *close neighborhood* $N_G[v]$ is $N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v \in A} N_G(v)$ and $N_G[A] = \bigcup_{v \in A} N_G[v]$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex x is said to be a *leaf* if $deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. Two leaves u and v are called the *duplicated leaves* in G if they are adjacent to the same support vertex. We denote by $L(G)$ and $U(G)$ the collections of all leaves and support vertices of G, respectively. We denote by $\tilde{L}(G)$ the collection of all duplicated leaves, and we denote by $\tilde{U}(G)$ the collection of all support vertices which are adjacent to some duplicated leaves. For two different sets A and B, written $A - B$ is the set of all elements of A that are not elements of B. For an edge $e \in E(G)$, the *deletion of e from G* is the graph $G - e$ obtained by removing the edge e. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. Denote C_n a cycle of order n. A graph G is called a *cactus graph* if it is a connected graph in which any two cycles have at most one vertex in common. For other undefined notions, the reader is referred to [2] for graph theory.

We need the following lemmas.

Lemma 2.1. *If G is a cactus graph with at least three vertices, then there exists a γ-set S of G such that $U(G) \subseteq S$.*

Proof. Let S be a γ-set S of G. If $U(G) \subseteq S$, then we are done. So we assume that $A = U(T) - S$, where $A \neq \emptyset$. Let $B = L(T) \cap N_G(A)$. Then $B \subseteq S$ and $|B| \geq |A|$. Let $S' = (S - B) \cup A$. Note that $N_G[B] \subset N_G[A]$. Then $N_G[S'] = V(G)$, so S' is a dominating set of G. Thus $|S| = \gamma(G) \leq |S'| = |S| - |B| + |A| \leq |S|$. Hence we obtain a γ-set S' of G such that $U(G) \subseteq S'$. □

Lemma 2.2. *If S is a γ_c-set of a cactus graph G, then $U(G) \subseteq S$.*

Proof. Suppose there exists a support vertex $v \notin S$ for some γ_c-set S in G. Let $L' = N_G(v) \cap L(G)$. Then $L' \subset S$. This contradicts that $\prec S \succ$ is connected. We complete the proof. □

Lemma 2.3. *If S is a (γ, γ_c)-set of a cactus graph G, then $U(G) \subseteq S$.*

Proof. It is a consequence of Lemma 2.1 and Lemma 2.2. □

Lemma 2.4. *Suppose G is a cactus graph and v is lying on some cycle C in G. Let S be a (γ, γ_c)-set of G. If $deg_G(v) \geq 3$, then $v \in S$.*

Proof. If v is a support vertex, by Lemma 2.3, then $v \in S$. So we assume that $v \notin U(G)$. Since G is a cactus graph and $deg_G(v) \geq 3$, $G - v$ is disconnected. Note that $\prec S \succ$ is connected, thus $v \in S$. We complete the proof. □

Lemma 2.5. *Suppose G is a cactus graph and C is a cycle of G. If S is a (γ, γ_c)-set of G and $A = \{v : v \in V(C), v \notin S\}$, then we have the following results.*
(i) *$A = \emptyset$ or $\prec A \succ$ is connected.*
(ii) *$deg_G(v) = 2$ for each $v \in A$.*
(iii) *$|A| \leq 2$.*

Proof. (i) Let $S' = S \cap V(C)$. Then $\prec S' \succ$ is connected, so $A = \emptyset$ or $\prec A \succ$ is connected. (ii) If $deg_G(u) \geq 3$ for some $u \in A$, by Lemma 2.4, then $u \in S$. This is a contradiction, so $deg_G(v) = 2$ for each $v \in A$. (iii) Suppose $|A| \geq 3$, say $A = \{v_1, v_2, v_3, \ldots\}$. By (ii), $deg_G(v_2) = 2$ and $N_G[v_2] = \{v_1, v_2, v_3\}$. Thus $S \cap N_G[v_2] = \emptyset$. This means that S is not a dominating set of G. It is a contradiction, so $|A| \leq 2$. □

Lemma 2.6. *Let G be a cactus graph and C be a cycle of G. Suppose S is a (γ, γ_c)-set of G and $A = \{v : v \in V(C), v \notin S\}$. If $|A| \leq 1$, then $v \in U(G)$ for each $v \in B$, where $B = V(C) - A$.*

Proof. We can see that $B = S \cap V(C)$. If $|A| \le 1$, then $|N_G[v] \cap S \cap V(C)| \ge 2$ for each $v \in V(C)$. Suppose there exists a vertex $u \in B$ such that $u \notin U(G)$. Let $S' = S - \{u\}$. If $deg_G(u) = 2$, then $|N_G(u) \cap S \cap V(C)| \ge 1$. So $u \in N_G[S']$ and $N_G[S'] = V(G)$, thus S' is a dominating set of G of cardinality $|S'| = |S| - 1$. This is a contradiction, so $deg_G(u) \ge 3$. Since $deg_G(u) \ge 3$ and $u \notin U(G)$, $G - u$ is disconnected and every component of $G - u$ has at least two vertices. Let $D = N_G(u) - V(C)$. Note that $\prec S \succ$ is connected, this means that $D \subset S$. Thus $u \in N_G[D]$. Then $u \in N_G[S']$ and $N_G[S'] = V(G)$, so S' is a dominating set of G of cardinality $|S'| = |S| - 1$. This is a contradiction again. Hence $v \in U(G)$ for each $v \in B$. We complete the proof. □

Lemma 2.7. [1] *For $k \ge 1$ and a tree T of order $|T| \ge 2k$, $\gamma_c(T) = \gamma(T) = k$ if and only if $V(T) = U(T) \cup L(T)$, where $|U(T)| = k$.*

3 Characterization

Xue-gang Chena, Liang Suna, Hua-ming Xing [3] characterized the cactus graphs for which the domination number is equal to the connected domination number. Their characterization is in terms of global properties of a construction. In this section, we provide a constructive characterization (Theorem 3.1) of the cactus graphs with equal domination and connected domination numbers.

For $m \ge 0$ and $k \ge 1$, let $\mathcal{G}(m,k)$ be the collection of all cactus graphs G which have exactly m cycles and $\gamma_c(G) = \gamma(G) = k$. In order to give a constructive characterization of $\mathcal{G}(m,k)$, we introduce four operations.

Operation O1. Assume $u, v \in U(G_i)$, where $uv \notin E(G_i)$, and the u-v path is unique in G_i. Add the edge uv.

Operation O2. Assume $u \in \tilde{L}(G_i)$, $v \in U(G_i)$, and the u-v path is unique in G_i. Add the edge uv.

Operation O3. Assume $u, v \in \tilde{L}(G_i)$ are adjacent to the same support vertices in G_i. Add the edge uv.

Operation O4. Assume $u \in L(G_i)$, $v \in \tilde{L}(G_i)$, and the u-v path is unique in G_i. Add the edge uv.

Let $\Psi(0,k)$ be the collection of the tree T which are $V(T) = U(T) \cap L(T)$ and $|U(T)| = k$. By Lemma 2.7, we obtain that $\Psi(0,k) = \mathcal{G}(0,k)$ for all $k \ge 1$. Suppose $\Psi(m,k)$, where $m \ge 1$ and $k \ge 1$, is the collection of the cactus graphs G, where G have exactly m cycles, that can be obtained from a sequence $G_0, G_1, \ldots, G_m = G$ of cactus graphs, where $G_i \in \Psi(i,k)$, and G_{i+1} is obtained recursively from G_i by one of the operation O1-O4.

Theorem 3.1. (*Characterization*) *For $m \geq 0$ and $k \geq 1$,*

$$\mathcal{G}(m,k) = \begin{cases} \Psi(m,k), & \text{if } m \neq 2\,; \\ \Psi(m,k) \cup \{C_4\}, & \text{if } m = 2, \end{cases}$$

where C_4 is the cycle of order four.

In order to prove the Theorem 3.1, we first prove the Lemma 3.2 and Lemma 3.3.

Lemma 3.2. *For $m \geq 0$ and $k \geq 1$, $\Psi(m,k) \subseteq \mathcal{G}(m,k)$.*

Proof. We prove this lemma by induction on $m \geq 0$. It's true for $m = 0$. Assume that it's true for $m-1$, where $m \geq 1$. Suppose $G \in \Psi(m,k)$ and C is a cycle of G. Since $G \in \Psi(m,k)$, G is obtained from some $G' \in \Psi(m-1,k)$ by one operation of O1-O4, say $G' = G - uv$. By induction hypothesis, $G' \in \mathcal{G}(m-1,k)$. Thus G is a cactus graph and G have exactly m cycles. Let S be a (γ,γ_c)-set of G'. By Lemma 2.3, $U(G') \subseteq S$. Note that $G' = G - uv$. So S is a dominating set and connected dominating set of G.
Claim. S is a (γ,γ_c)-set of G. We consider four cases.
<u>Case 1</u>. G is obtained from G' by Operation O1. Then $u, v \in U(G')$ and $U(G) = U(G')$. So $u, v \in U(G)$, by Lemma 2.3, u and v are in every (γ,γ_c)-set of G. Note that $G' = G - uv$, hence S is a (γ,γ_c)-set of G.
<u>Case 2</u>. G is obtained from G' by Operation O2. Let $N_G(u) \cap U(G') = \{u'\}$. Then $u' \in \widetilde{U}(G')$, $v \in U(G')$ and $U(G) = U(G')$. So $u', v \in U(G)$, by Lemma 2.3, u and v are in every (γ,γ_c)-set of G. Note that $G' = G - uv$, hence S is a (γ,γ_c)-set of G.
<u>Case 3</u>. G is obtained from G' by Operation O3. Then u and v are duplicated leaves adjacent to the same support vertex w in G'. Then we can see that $U(G') = U(G) \cup \{w\}$ and $w \in S$. Thus w is in in every (γ,γ_c)-set of G. Note that $G' = G - uv$, hence S is a (γ,γ_c)-set of G.
<u>Case 4</u>. G is obtained from G' by Operation O4. Let $N_G(u) \cap U(G') = \{u'\}$ and $N_G(v) \cap U(G') = \{v'\}$, where $u' \in \widetilde{U}(G')$. Thus $u', v' \in U(G')$, by Lemma 2.3, $u', v' \in S$. Note that $u' \in \widetilde{U}(G')$, so $u \in U(G)$. By Lemma 2.3, $u' \in S$ and $u \notin S$. Since $N_G(v) = \{u, v'\}$ and $u \notin S$, hence S is a (γ,γ_c)-set of G.

By Case 1, Case 2, Case 3 and Case 4, S is a (γ,γ_c)-set of G. Hence G is a cactus graph having exactly m cycles and $\gamma_c(G) = \gamma(G) = |S| = k$. That is $G \in \mathcal{G}(m,k)$. So it's true for m. We complete the proof. □

Lemma 3.3. *If $G \in \mathcal{G}(m,k)$ and $G \neq C_4$, where $m \geq 0$ and $k \geq 1$, then $G \in \Psi(m,k)$.*

Proof. Note that C_4 is not a tree, so it's true for $m = 0$. We prove this lemma by contradiction, assume it's not true for some $m' \geq 1$. Suppose there exists a graph $G \in \mathcal{G}(m^*, k)$, $G \notin \Psi(m^*, k)$ and $G \neq C_4$ such that m^* is as small as possible. Then $m^* \geq 1$. Assume that $C : v_1, v_2, \ldots, v_n, v_1$ is a cycle of G. Let S be a (γ, γ_c)-set of G and $A = \{v : v \in V(C), v \notin S\}$. By Lemma 2.5, $|A| \leq 2$ and $deg_G(v) = 2$ for each $v \in A$. We consider three cases.

Case 1. $|A| = 0$. By Lemma 2.6, $v_i \in U(G)$ for each i. Let $G' = G - v_1v_2$ be the deletion of the edge v_1v_2 from G. Then $v_i \in U(G')$ and $v_i \in S$ for all i, by Lemma 2.3, so S is a (γ, γ_c)-set of G'. Note that G' is a cactus graph with $m^* - 1$ cycles and $\gamma_c(G') = \gamma(G') = |S| = k$. That is $G' \in \mathcal{G}(m^* - 1, k)$, by the hypothesis, $G' \in \Psi(m^* - 1, k)$. Note that $v_1, v_2 \in U(G')$. Hence G is obtained from $G' \in \Psi(m^* - 1, k)$ by the Operation O1. Thus $G \in \Psi(m^*, k)$, this is a contradiction.

Case 2. $|A| = 1$, say $A = \{v_1\}$. By Lemma 2.6, $v_i \in U(G)$ for all $i \neq 1$. Let $G' = G - v_1v_2$ be the deletion of the edge v_1v_2 from G. Then $v_i \in U(G')$ and $v_i \in S$ for all $i \neq 1$, by Lemma 2.3, so S is a (γ, γ_c)-set of G'. Note that G' is a cactus graph with $m^* - 1$ cycles and $\gamma_c(G') = \gamma(G') = |S| = k$. That is $G' \in \mathcal{G}(m^* - 1, k)$, by the hypothesis, $G' \in \Psi(m^* - 1, k)$. Note that $v_1 \in \widetilde{L}(G')$ and $v_2 \in U(G')$. Hence G is obtained from $G' \in \Psi(m^* - 1, k)$ by the Operation O2. Thus $G \in \Psi(m^*, k)$, this is a contradiction.

Case 3. $|A| = 2$, say $A = \{v_1, v_2\}$. By Lemma 2.6, $deg_G(v_1) = deg_G(v_2) = 2$. Let $G' = G - v_1v_2$ be the deletion of the edge v_1v_2 from G. Note that G' is a cactus graph with exactly $m^* - 1$ cycles. If $|C| = 3$, then v_1 and v_2 are duplicated leaves adjacent to the vertex v_3 in G'. Then $v_3 \in U(G')$ and $v_3 \in S$, so S is a (γ, γ_c)-set of G', thus $\gamma_c(G') = \gamma(G') = |S| = k$. That is $G' \in \mathcal{G}(m^* - 1, k)$, by the hypothesis, $G' \in \Psi(m^* - 1, k)$. Note that v_1 and v_2 are duplicated leaves adjacent to the vertex v_3 in G'. Hence G is obtained from $G' \in \Psi(m^* - 1, k)$ by the Operation O3, thus $G \in \Psi(m^*, k)$. This is a contradiction, so $|C| \geq 4$. We consider two subcases.

<u>Case 3.1</u>. $v_3 \in U(G)$ or $v_n \in U(G)$, say $v_n \in U(G)$. Then $v_i \in U(G')$ and $v_i \in S$ for all $i \neq 1, 2$, by Lemma 2.3, so S be a (γ, γ_c)-set of G'. Then $v_i \in U(G')$ and $v_i \in S$ for all $i \neq 1, 2$, by Lemma 2.3, so S is a (γ, γ_c)-set of G'. Thus $\gamma_c(G') = \gamma(G') = |S| = k$. That is $G' \in \mathcal{G}(m^* - 1, k)$, by the hypothesis, $G' \in \Psi(m^* - 1, k)$. Note that $v_1 \in \widetilde{L}(G')$ and $v_2 \in L(G')$. Hence G is obtained from $G' \in \Psi(m^* - 1, k)$ by the Operation O4. Thus $G \in \Psi(m^*, k)$, this is a contradiction.

<u>Case 3.2</u>. $v_3 \notin U(G)$ and $v_n \notin U(G)$. Let $S' = S - \{v_3, v_n\}$. If $|C| \geq 5$, then $v_4, \ldots, v_{n-1} \in S'$ and $N_G[S' \cup \{v_1\}] = V(G)$. So $\gamma(G) \leq |S' \cup \{v_1\}| = |S| - 1 = k - 1$. This is a contradiction, thus $|C| = 4$. If $deg_G(v_3) \geq 3$, then $N_G[S' \cup \{v_1\}] = V(G)$. So $\gamma(G) \leq |S' \cup \{v_1\}| = |S| - 1 = k - 1$. This is a contradiction, thus $deg_G(v_3) = 2$. Similarly, $deg_G(v_n) = 2$. That is $G = C_4$, this is a contradiction again.

By Case 1, Case 2 and Case 3, it's a contradiction. We complete the proof. $\square$

As an immediate consequence of Lemma 3.2 and Lemma 3.3, we obtain the Theorem 3.1.

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