Heronian Mean Labeling of

Disconnected Graphs

S.S. Sandhya
Department of Mathematics
Sree Ayyappa College for Women
Chunkankadai – 629003, Tamilnadu, India

E. Ebin Raja Merly
Department of Mathematics
Nesamony Memorial Christian College
Marthandam – 629165, Tamilnadu, India

S.D. Deepa
Nesamony Memorial Christian College
Marthandam – 629165, Tamilnadu, India

Abstract

In this paper, we contribute some new results for Heronian Mean labeling of graphs. We prove that disconnected Heronian Mean Graphs are Heronian Mean Graphs. We use some standard graphs to derive the results for disconnected graphs.

Mathematics Subject Classification: 05C78

Keywords: Graph, Heronian Mean Graph, Path, Cycle, Comb, Ladder, Triangular Snake, Quadrilateral Snake
1. Introduction

By a graph we mean a finite undirected graph without loops or parallel edges. For all detailed survey of graph labeling, we refer to J.A. Gallian [1]. For all other standard terminology and notations we follow Harary [2]. The concept of Mean labeling has been introduced by S. Somasundaram and R. Ponraj [3] in 2004. S. Somasundaram and S.S. Sandhya introduced Harmonic mean labeling [4] in 2012. Motivated by the above works we introduced a new type of labeling called Heronian Mean Labeling in [5].

In this paper we investigate the Heronian Mean Labeling of some disconnected graphs. We will provide brief summary of definitions and other information which are necessary for our present investigation.

A Path $P_n$ is a walk in which all the vertices are distinct. A Cycle $C_n$ is a Closed Path. The graph obtained by joining a single pendant edge to each vertex of a Path is called a Comb. The corona $G_1 \circ G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ (which has $P_1$ vertices) and $P_1$ copies of $G_2$ and then joining the $i$th vertex of $G_1$ to every vertices in the $i$th copy of $G_2$. The graph $C_n \circ K_1$ is called crown. The Ladder $L_n$ is the product graph $P_2 \times P_n$. A Triangular Snake $T_n$ is obtained from a path $u_1,u_2,\ldots,u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $1 \leq i \leq n-1$. That is every edge of a path is replaced by a triangle $C_3$. A Quadrilateral Snake $Q_n$ is obtained from a path $u_1,u_2,\ldots,u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ respectively and then joining $v_i$ and $w_i$. That is every edge of a path is replaced by a cycle $C_4$.

Definition 1.1:
A graph $G=(V,E)$ with $p$ vertices and $q$ edges is said to be a Heronian Mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\ldots,q+1$ in such a way that when each edge $e=uv$ is labeled with,

$$f(e=uv) = \frac{f(u) + \sqrt{f(u)f(v) + f(v)}}{3} \quad (\text{OR}) \quad \frac{f(u) + \sqrt{f(u)f(v)}}{3}$$

Then the edge labels are distinct. In this case $f$ is called a Heronian Mean labeling of $G$.

Theorem 1.2: Any Path $P_n$ is a Heronian mean graph.

Theorem 1.3: Any Comb $P_n \circ K_1$ is a Heronian mean graph.

Theorem 1.4: Any Cycle $C_n$ is a Heronian mean graph.

Theorem 1.5: Crown, $C_n \circ K_1$ is a Heronian mean graph for all $n \geq 3$.

Theorem 1.6: Any Triangular Snake $T_n$ is a Heronian mean graph.

Theorem 1.7: Any Quadrilateral Snake $Q_n$ is a Heronian mean graph.

Theorem 1.8: Any Ladder $L_n$ is a Heronian mean graph.
2. Main Results

**Theorem 2.1**

$P_m \cup P_n$ is a Heronian mean graph.

**Proof:**

Let $P_m$ be a path $u_1u_2u_3 \ldots u_m$ and $P_n$ be a path $v_1v_2v_3 \ldots v_n$ respectively.

Define a function $f: V(P_m \cup P_n) \rightarrow \{1,2,3,\ldots,q+1\}$ by $f(u_i) = i, 1 \leq i \leq m$.

$$f(v_i) = m + i, 1 \leq i \leq n.$$ 

Edges are labeled with $f(u_iu_{i+1}) = i, 1 \leq i \leq m$,

$$f(v_i v_{i+1}) = i, 1 \leq i \leq n.$$ 

Hence $P_m \cup P_n$ is a Heronian mean graph.

**Example 2.2:** A Heronian mean labeling of $P_5 \cup P_4$ is given below.

![Figure 1]

**Theorem 2.3**

$C_m \cup P_n$ is a Heronian mean graph for $m \geq 3$ and $n > 1$.

**Proof:**

Let $C_m$ be the cycle $u_1u_2u_3 \ldots u_m u_1$ and $P_n$ be a path $v_1v_2v_3 \ldots v_n$ respectively.

Define a function $f: V(C_m \cup P_n) \rightarrow \{1,2,3,\ldots,q+1\}$ by $f(u_i) = i, 1 \leq i \leq m$.

$$f(v_i) = m + i, 1 \leq i \leq n.$$ 

Edges of $C_m$ are labeled by $f(u_iu_{i+1}) = i + 1, 1 \leq i \leq m$,

$$f(u_m u_1) = 1$$

Edges of $P_n$ are labeled by $\{m + 1, m + 2, \ldots, m + n - 1\}$.

Hence $C_m \cup P_n$ is a Heronian mean graph if $m \geq 3$ and $n > 1$.

**Example 2.4:** A Heronian mean labeling of $C_5 \cup P_4$ is given below.
Theorem: 2.5
\[ C_m \cup (P_n \odot K_1) \text{ is a Heronian mean graph for } m \geq 3 \text{ and } n > 1. \]

Proof:
Let \( C_m \) be the cycle \( u_1u_2u_3 \ldots u_mu_1 \) and \( P_n \odot K_1 \) be a graph obtained from a path \( v_1v_2 \ldots v_n \) by joining the vertex \( v_i \) to pendant vertices \( w_i \) respectively.
Define a function, \( f: V(C_m \cup (P_n \odot K_1)) \rightarrow \{1, 2, 3, \ldots, q + 1\} \) by \( f(u_i) = i, \ 1 \leq i \leq m \)
\[ f(v_i) = m + (2i - 1), \ 1 \leq i \leq n \]
\[ f(w_i) = m + 2i, \ 1 \leq i \leq n. \]

We get distinct edge labels for \( C_m \).
Edges of \( P_n \odot K_1 \) are labeled by \( \{m + 1, m + 2, \ldots, m + 2n - 1\} \).
Hence \( C_m \cup (P_n \odot K_1) \) is a Heronian mean graph if \( m \geq 3 \text{ and } n > 1. \)

Example 2.6: A Heronian mean labeling of \( C_4 \cup (P_5 \odot K_1) \) is given below.

Theorem: 2.7
\[ (C_m \odot K_1) \cup P_n \text{ is a Heronian mean graph for } m \geq 3 \text{ and } n > 1. \]

Proof:
Let \( C_m \odot K_1 \) be the cycle \( u_1u_2u_3 \ldots u_mu_1 \) by joining the vertex \( u_i \) to pendant vertices \( v_i \) and let \( P_n \) be a path \( w_1w_2w_3 \ldots w_n \) respectively.
Define a function, \( f: V((C_m \odot K_1) \cup P_n) \rightarrow \{1, 2, 3, \ldots, q + 1\} \) by \( f(u_i) = 2i, \ 1 \leq i \leq m \)

Figure: 2

Figure: 3
Heronian mean labeling of disconnected graphs

\[ f(v_i) = 2i - 1, \quad 1 \leq i \leq m \]
\[ f(w_i) = m + i, \quad 1 \leq i \leq n. \]

We get distinct edge labels for \( C_m \odot K_1 \).

Edges of \( P_n \) are labeled by \( \{2m + 1, 2m + 2, \ldots, 2m + n - 1\} \).

Hence \( (C_m \odot K_1) \cup P_n \) is a Heronian mean graph if \( m \geq 3 \) and \( n > 1 \).

**Example 2.8**: A Heronian mean labeling of \( (C_3 \odot K_1) \cup P_6 \) is given below.

![Figure 4]

**Theorem 2.9**

\( C_m \cup (C_n \odot K_1) \) is a Heronian mean graph for \( m \geq 3 \) and \( n \geq 3 \).

**Proof:**

Let \( C_m \) be the cycle \( u_1u_2u_3 \ldots u_mu_1 \) and \( C_n \odot K_1 \) be a graph obtained from a path \( v_1v_2 \ldots v_n \) by joining the vertex \( v_i \) to pendant vertices \( w_i \) respectively.

Define a function, \( f : V(C_m \cup (C_n \odot K_1)) \rightarrow \{1, 2, 3, \ldots, q + 1\} \) by

\[ f(u_i) = i, \quad 1 \leq i \leq m \]
\[ f(v_i) = m + 2i, \quad 1 \leq i \leq n \]
\[ f(w_i) = m + (2i - 1), \quad 1 \leq i \leq n. \]

We get distinct edge labels for \( C_m \) and \( C_n \odot K_1 \).

Hence \( C_m \cup (C_n \odot K_1) \) is a Heronian mean graph if \( m \geq 3 \) and \( n \geq 3 \).

**Example 2.10**: A Heronian mean labeling of \( C_4 \cup (C_3 \odot K_1) \) is given below.

![Figure 5]
Theorem 2.11

$C_m \cup L_n$ is a Heronian mean graph for $m \geq 3$ and $n > 1$.

Proof:

Let $C_m$ be the cycle $u_1u_2u_3 \ldots u_mu_1$ and $L_n$ be a ladder connecting two paths $v_1v_2 \ldots v_n$ and $w_1w_2 \ldots w_n$.

Define a function, $f: V(C_m \cup L_n) \rightarrow \{1,2,3, \ldots, q + 1\}$ by $f(u_i) = i$, $1 \leq i \leq m$

\[ f(v_i) = m + (3i - 2), \quad 1 \leq i \leq n \]
\[ f(w_i) = m + (3i - 1), \quad 1 \leq i \leq n. \]

We get distinct edge labels for $C_m$.

Edges of $L_n$ are labeled by $\{m + 1, m + 2, \ldots, m + 3n - 2\}$.

Hence $C_m \cup L_n$ is a Heronian mean graph if $m \geq 3$ and $n > 1$.

Example 2.12: A Heronian mean labeling of $C_4 \cup L_5$ is given below.

\[ \text{Figure: 6} \]

Theorem 2.13

$(C_m \odot K_1) \cup L_n$ is a Heronian mean graph for $m \geq 3$ and $n > 1$.

Proof:

Let $C_m \odot K_1$ be a graph obtained from the cycle $u_1u_2u_3 \ldots u_mu_1$ by joining the vertex $u_i$ to pendant vertices $v_i$ and let $L_n$ be a ladder connecting two paths $x_1x_2 \ldots x_n$ and $y_1y_2 \ldots y_n$ respectively.

Define a function, $f: V((C_m \odot K_1) \cup L_n) \rightarrow \{1,2,3, \ldots, q + 1\}$ by $f(u_i) = 2i$, $1 \leq i \leq m$

\[ f(v_i) = 2i - 1, \quad 1 \leq i \leq m \]
\[ f(x_i) = m + (3i - 2), \quad 1 \leq i \leq n. \]
\[ f(y_i) = m + (3i - 1), \quad 1 \leq i \leq n. \]

We get distinct edge labels for $C_m \odot K_1$.

Edges of $L_n$ are labeled by $\{2m + 1,2m + 2, \ldots, 2m + 3n - 2\}$.

Hence $(C_m \odot K_1) \cup L_n$ is a Heronian mean graph if $m \geq 3$ and $n > 1$. 
**Example 2.14:** A Heronian mean labeling of \((C_3 \circ K_1) \cup L_5\) is given below.

![Graph 1](image1)

**Figure: 7**

**Theorem 2.15**

\((C_m \circ K_1) \cup (P_n \circ K_1)\) is a Heronian mean graph for \(m \geq 3\) and \(n > 1\).

**Proof:**

Let \(C_m \circ K_1\) be a graph obtained from the cycle \(u_1u_2u_3\ldots u_mu_1\) by joining the vertex \(u_i\) to pendant vertices \(v_i\) and let \(P_n \circ K_1\) be a graph obtained from a path \(x_1x_2\ldots x_n\) by joining the vertex \(x_i\) to pendant vertices \(y_i\) respectively.

Define a function, \(f: V((C_m \circ K_1) \cup (P_n \circ K_1)) \rightarrow \{1, 2, 3, \ldots q + 1\}\) by

\[
f(u_i) = 2i, \quad 1 \leq i \leq m \\
f(v_i) = 2i - 1, \quad 1 \leq i \leq m \\
f(x_i) = m + (2i - 1), \quad 1 \leq i \leq n. \\
f(y_i) = m + 2i, \quad 1 \leq i \leq n
\]

We get distinct edge labels for \(C_m \circ K_1\).

Edges of \(P_n \circ K_1\) are labeled by \(\{2m + 1, 2m + 2, \ldots 2m + 2n - 1\}\).

Hence \((C_m \circ K_1) \cup (P_n \circ K_1)\) is a Heronian mean graph if \(m \geq 3\) and \(n > 1\).

**Example 2.16:** A Heronian mean labeling of \((C_3 \circ K_1) \cup (P_5 \circ K_1)\) is given below.

![Graph 2](image2)

**Figure: 8**
3. Conclusion

The Study of labeled graph is important due to its diversified applications. It is very interesting to investigate disconnected graphs which admit Heronian Mean Labeling. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. It is possible to investigate similar results for several other graphs.

Acknowledgements. The authors are thankful to the referee for their valuable comments and suggestions.

References


Received: August 17, 2016; Published: July 17, 2017