On a Diophantine Equation$^1$

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Abstract

In this note, we mainly obtain the equation $x^{2m} - y^n = z^2$ have finite positive integer solutions $(x, y, z, m, n)$ satisfying $x > y$ be two consecutive primes.

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1 Introduction and main results

In 1844, Catalan proposed the following conjecture.

Conjecture 1.1 The only two consecutive numbers in the sequence of perfect powers of natural numbers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, \ldots$

are 8 and 9.

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Between 2000 and 2004, Mihăilescu [9], [10] proved this conjecture is true. Before this, there are many efforts on the Catalan Conjecture and a series of such equations were studied. As a general case, the Diophantine equation

$$ax^m - by^n = c, \quad a, b, c, x, y, m, n \in \mathbb{Z}$$

was extensive studied by many experts. One can see [3], [8], [5], [6] for more detail.

In this note, we consider the following equation

$$x^m - y^n = z^2. \tag{1.1}$$

In 2002, Le [7] showed that the equation (1.1) has no solution for $y = 2$ and $2|n$. In 2008, Bérczes and Pink [2] gave the solution about the equation (1.1) in the case that $y = p$, $2|n$, and $2 \leq p < 100$. In 2016, Ventullo [13] gave some examples to the equation (1.1) in the case that $x > y$ are two consecutive primes.

We continue to study the equation (1.1) as in [13] which consider the case that $x > y$ are two consecutive primes. It is obviously that $(m, n, z) = (0, 0, 0)$ is a solution for any given consecutive primes $p, q$ to the equation (1.1). We call this solution as trivial solution. It is nature to ask the question that does there exists consecutive primes $p, q$ such that the equation (1.1) has only the trivial solution? Actually, we have the following result:

**Theorem 1.1** There are infinitely many consecutive primes $p$ and $q$ ($p > q$) such that the equation

$$p^m - q^n = z^2$$

has only the trivial solution.

Another question is that dose it have finite solutions if the equation (1.1) have non-trivial solutions. In fact, we obtain:

**Theorem 1.2** Let $p, q$ be two primes. Then the equation

$$p^{2m} - q^n = z^2$$

has at most one non-trivial solution $(m, n, z)$ in natural number except $q = 2$.

**Theorem 1.3** There are only finite solutions $(x, y, z, m, n)$ to the equation

$$x^{2m} - y^n = z^2$$

in natural number such that $x > y$ be two consecutive primes.
2 Some lemmas

In this section, we give some examples and some useful lemmas.

**Lemma 2.1** ([12]) Let $A$ be a discrete valuation ring, and let $x_i$ be elements of the field of fractions of $A$ such that $v(x_i) > v(x_1)$ for $i \geq 2$. One then has $\sum_{i=1}^{n} x_i \neq 0$.

**Lemma 2.2** Let $p$ be a prime and $n$ a natural number. Then $\text{ord}_p(n) \leq \log n / \log p$. Moreover, if $n > 2$, then $\text{ord}_p(n(n-1)) < n - 1$.

**Lemma 2.3** Let $m, n$ be two positive integers. Then
\[
\sum_{m=0}^{n} \binom{2n+1}{2m+1} (-1)^{m+1} 2^m \neq -1.
\]

**Proof:** Firstly, we assume that
\[
\sum_{m=0}^{n} \binom{2n+1}{2m+1} (-1)^{m+1} 2^m = -1.
\]
Then
\[
-2n + \binom{2n+1}{3} 2 - \binom{2n+1}{5} 2^2 + \sum_{m=3}^{n} \binom{2n+1}{2m+1} (-1)^{m+1} 2^m = 0.
\]
By Lemma 2.2, for $m = 3, 4, \ldots, n$,
\[
\text{ord}_2 \left( \frac{\binom{2n+1}{2m+1} (-1)^{m+1} 2^m}{\binom{2n+1}{5} 2^2} \right) = m - 1 + \text{ord}_2 \left( \binom{2n-4}{2m-4} - \text{ord}_2(m(m-1)) \right)
\]
\[
> \text{ord}_2 \left( \binom{2n-4}{2m-4} \right) \geq 0.
\]
Then we obtain
\[
\text{ord}_2 \left( \binom{2n+1}{2m+1} (-1)^{m+1} 2^m \right) > \text{ord}_2 \left( -\binom{2n+1}{5} 2^2 \right)
\]
\[
\geq \text{ord}_2 \left( \binom{2n+1}{3} 2 \right) = \text{ord}_2(-2n).
\]
On the other hand,
\[
\text{ord}_2 \left( \binom{2n+1}{3} 2 - 2n \right) = \text{ord}_2(8n(n-1)) > \text{ord}_2(2n(n-1)) = \text{ord}_2 \left( -\binom{2n+1}{5} 2^2 \right).
\]
Then by Lemma 2.1, the equation is impossible. Thus the proof of Lemma 2.3 is finished.
Proposition 2.1 The only pairs of natural numbers \((x, y)\) such that \(3^x - 2^y\) is a perfect square are \((0, 0), (1, 1), (2, 3), (3, 1), (4, 5)\).

Proof: Let
\[
3^x - 2^y = z^2 \tag{2.1}
\]
Clearly, if \(x < 5\), then the integer solutions are \((x, y, z) = (0, 0, 0), (1, 1, 1), (2, 3, 1), (3, 1, 5), (4, 5, 7)\). We will prove that there are no solution in natural numbers for any \(x \geq 5\).

If \((x, y, z)\) is a solution of the equation 2.1, then
\[-2^y \equiv z^2 \pmod{3}.
\]
So \(y\) is odd. If \(y = 1\), then \(3^x - 2 = z^2\). Clearly, \(x\) is odd, otherwise \(z^2 \equiv -1 \pmod{4}\), which is impossible. In the ring of integers \(\mathbb{Z}[\sqrt{-2}]\), we have
\[3^x = (z - \sqrt{-2})(z + \sqrt{-2}).\]

\(n - \sqrt{-2}\) and \(n + \sqrt{-2}\) is coprime in \(\mathbb{Z}[\sqrt{-2}]\). Otherwise, let \(d = \gcd(n - \sqrt{-2}, n + \sqrt{-2})\). Then \(|N(d)| > 1\) and \(d|2\sqrt{-2}\), so \(N(d)|8\), which impossible since \(N(d)|9\). We have \(3 = (1 - \sqrt{-2})(1 + \sqrt{-2})\), so we have \(n - \sqrt{-2} = \pm(1 - \sqrt{-2})^x\) or \(n + \sqrt{-2} = \pm(1 + \sqrt{-2})^x\). Consider the imaginary part of equation \(n - \sqrt{-2} = (1 - \sqrt{-2})^x\) or \(n + \sqrt{-2} = (1 + \sqrt{-2})^x\). We obtain
\[-1 = \sum_{k=1, k \text{ odd}}^{x} \binom{x}{k} (-1)^{k+1} 2^{k+1}.
\]
This is impossible by Lemma 2.3. Then we have \(y \geq 2\). Hence \(3^x \equiv z^2 \pmod{4}\). So \(x\) is even. Therefore, equation 2.1 becomes \((3^x - z)(3^x + z) = 2^y\).

It follows that \(\gcd(3^x - z, 3^x + z) = \gcd(3^x - z, 2 \cdot 3^x) = 2\). Thus, \(3^x - z = 2\), and \(3^x - z = 2\). So we get
\[3^x - 2^y - 2 = 1.
\]
By Catalan’s conjecture, the only positive integer solution of this equation is \((x, y) = (4, 5)\), contradiction. Thus the proof of Proposition 2.1 is finished.

Let \(\alpha\) be an algebraic number with minimal polynomial
\[f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{Z}[x],\]
where \(a_0 > 0\). Then we can write \(f(x) = a_0 \prod_{i=1}^{d} (x - \sigma_i \alpha)\), where \(\sigma_1 \alpha, \cdots, \sigma_d \alpha\) are all conjugates of \(\alpha\). Let
\[h(\alpha) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \max\{1, |\sigma_i \alpha|\} \right)\]
be the absolute logarithmic height of \(\alpha\).
Lemma 2.4 (See [1]) Denote by \(\alpha_1, \alpha_2, \ldots, \alpha_n\) algebraic numbers, not 0 or 1, by \(\log \alpha_1, \log \alpha_2, \ldots, \log \alpha_n\) determinations of their logarithms, by \(D\) the degree over \(\mathbb{Q}\) of the number field \(K = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)\), and by \(b_1, b_2, \ldots, b_n\) rational integers. Define \(B = \max\{|b_1|, |b_2|, \ldots, |b_n|\}\), and \(A_i = \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\}\) for all \(1 \leq i \leq n\), where \(h(\alpha)\) denotes the absolute logarithmic height of \(\alpha\). Assume that the number \(\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + \ldots + b_n \log \alpha_n\) does not vanish, then

\[|\Lambda| \geq \exp\{-C(n, \lambda)D^2A_1A_2 \ldots A_n \log(eD) \log(eB)\},\]

where \(\lambda = 1\) if \(K \subseteq \mathbb{R}\) and \(\lambda = 2\) otherwise and

\[C(n, \lambda) = \left\{ \frac{en}{\lambda} \right\}^{30n+3}n^{3.5}2^{6n+10}\].

Lemma 2.5 Let \(p_n\) denote the \(n\)-th prime. Then

1. \(p_n \leq n \log n + n \log \log n\) for \(n \geq 6\).
2. \(p_n \geq n \log n + n(\log \log n - 1)\) for \(n \geq 2\).


Lemma 2.6 Let \(p, q\) be two odd primes. If \((m_0, n_0)\) is a solution of

\[2p^m - q^n = 1\]

with \(m_0, n_0 > 0\), then \(n_0 = 2^s\) for some nonnegative integer \(s\).

Proof: Let \((m_0, n_0)\) be a solution of \(2p^m - q^n = 1\). Suppose that there exists an odd prime \(l\) dividing \(n_0\), we have \(n_0 = kl\) for some integer \(k \geq 1\). Then

\[2p^{m_0} = q^{n_0} + 1 = q^k + 1 = (q^k + 1)(q^{k(l-1)} - q^{k(l-2)} + \ldots + 1).\]

Hence we have

\[\frac{q^k + 1}{q^k} = q^{k(l-1)} - q^{k(l-2)} + \ldots + 1 > l.\] (2.2)

and \(q^k + 1 = 2p^{m_1}\), for some \(1 \leq m_1 < m_0\). Therefore,

\[p^{m_0 - m_1} = \frac{q^k + 1}{q^k} = \frac{(2p^{m_1} - 1)^l + 1}{2p^{m_1}} = \sum_{i=1}^{l} \binom{l}{i} (2p^{m_1})^{i-1}(-1)^{l-i}.\] (2.3)

Modulo \(p\) in both side of the equation (2.3), we obtain

\[0 \equiv \sum_{i=1}^{l} \binom{l}{i} (2p^{m_1})^{i-1}(-1)^{l-i} \equiv l \pmod{p}.\]
This force \( l = p \). Then by equation (2.2) we have \( p^{m_0 - m_1} > p \).

On the other hand, modulo \( p^2 \) in both side of the equation (2.3), we have

\[
p^{m_0 - m_1} = \sum_{i=1}^{l} \binom{l}{i} (2p^{m_1})^{i-1}(-1)^{l-i} \equiv p \pmod{p^2}.
\]

This force \( p^{n_0 - m_1} = p \), contradiction. So \( n_0 = 2^s \) for some integer \( s \).

**Lemma 2.7** For any fixed integer \( n > 0 \), the equation \( 2x^m - y^n = 1 \) has finite solutions \( (x, y, m) \in \mathbb{Z}_{>0} \) such that \( x > y \) are two consecutive primes.

## 3 Proof of main results

**Proof of Theorem 1.1**

We will proof that if \( p, q \) satisfy the condition that

\[
p \equiv 3 \pmod{4}, \quad q \equiv 1 \pmod{4}
\]

then \( p^x - q^y = z^2 \) has no nontrivial integer solution. Otherwise, let \((x, y, z)\) is a solution. Then \( p^x - q^y \equiv 0 \pmod{4} \), so \( 2|x \). Therefore, the equation becomes

\[
(p^{\frac{x}{2}} - z)(p^{\frac{x}{2}} + z) = q^y.
\]

It follows that \( p^{\frac{x}{2}} - z = 1 \) and \( p^{\frac{x}{2}} + z = q^y \), since \( \gcd(p^{\frac{x}{2}} - z, p^{\frac{x}{2}} + z) = 1 \). So we get

\[
2 \cdot p^{\frac{x}{2}} = 1 + q^y.
\]

In the other hand, Modulo \( p \) in the equation, we obtain \(-q^y \equiv z^2 \pmod{p} \). So \( y \) must be odd, since \((\frac{-1}{p}) = -1 \). Hence we have

\[
2 \cdot p^{\frac{x}{2}} = 1 + q^y = (1 + q)(q^{y-1} - q^{y-2} + \ldots + 1),
\]

so \( 2p|(1 + q) \), it is contradict with \( p > q \).

At last, there are infinitely many consecutive primes \( p \) and \( q \) \((p > q)\) that satisfy the condition 3.1. This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2**

By Proposition 2.1, we see that there are 3 solutions when \((p, q) = (3, 2)\).

In the following, we suppose that \( q > 2 \). Assume the assertion is false, that is, that there exists two different non-trivial solutions \((x_1, y_1), (x_2, y_2)\) such that \( x_2 > x_1 \geq 1 \). Then

\[
\begin{cases}
p^{2x_1} - q^{y_1} = z_1^2, \\
p^{2x_2} - q^{y_2} = z_2^2.
\end{cases}
\]
So we obtain
\[(p^{x_1} - z_1)(p^{x_1} - z_1) = q^{y_1}.\]

It follows that \(p^{x_1} - z_1 = q^a\) and \(p^{x_1} + z_1 = q^b\), where \(a, b \in \mathbb{N}\) and \(a + b = y_1\). Thus \(gcd(p^{x_1} - z_1, p^{x_1} + z_1) = q^a\). Hence \(q^a \mid 2p^{x_1}\). So we get \(a = 0\) because \(q > 2\). Then we obtain
\[2p^{x_1} - q^{y_1} = 1.\]

Similarly, we have
\[2p^{x_2} - q^{y_2} = 1.\]

If \(p = 2\). Then we have \(2^{x_1+1} - 1 = q^u\), \(i = 1, 2\). Hence, \(x_1 + 1\) and \(x_2 + 1\) are primes. Thus, \(gcd(2^{x_1+1} - 1, 2^{x_2+1} - 1) = 2^{gcd(x_1+1, x_2+1)} - 1 = 1\), which is impossible.

If \(p > 2\). Then by Lemma 2.6, we obtain that there exist an integer \(s > 0\) such that \(y_2 = 2^s y_1\). Thus
\[2p^{x_2} = 1 + q^{y_2} = 1 + (2p^{x_1} - 1)^{2^s}.\]

Modulo \(p\) in both side of this equation, we have \(0 \equiv 2 \pmod{p}\), a contradiction. This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3**
Let \(p > q\) be two consecutive primes that bigger than 2. Then by the proof of Theorem 1.2, we have \(p^{2m} - q^n = z^2\) is equal to \(2p^{m} - q^n = 1\). Hence, it is enough to prove that the equation
\[2x^m - y^n = 1 \quad (3.2)\]
only have finite solutions \((x, y, m, n)\) in natural number such that \(x > y\) be two consecutive primes.

By Lemma 2.7, the equation (3.2) only have finite solutions for \(n < 16\). Hence we consider the case \(n \geq 16\). Let \(p_k\) be the \(k\)-th prime, \(m, n\) positive integers, and let
\[S_0 = \{(p_{k+1}, p_k, m, n) \mid 2p^m_{k+1} - p_k^n = 1, \ n \geq 16\}.\]

We shall show that the set \(S_0\) finite. Set
\[S_1 = \{(p_{k+1}, p_k, m, n) \mid k + 1 > e^{n^{3/4}}\},\]
\[S_2 = \{(p_{k+1}, p_k, m, n) \mid k + 1 < e^{n^{3/4}}\}.\]

Then it’s enough to prove that the sets \(S_0 \cap S_1\) and \(S_0 \cap S_2\) are all finite.
Let \((p_{k+1}, p_k, m, n) \in S_0 \cap S_1\). Then we have \(p_{k+1} > p_k \geq k + 1 > e^{n^{3/4}}\). For \(n \geq 16\), by Lemma 2.5, we have

\[
\varepsilon = \frac{p_{k+1} - p_k}{p_k} < \frac{(k + 1) \log(k + 1) + (k + 1) \log \log(k + 1) - k \log -k(\log \log k - 1)}{p_k} < \frac{2 \log(k + 1) + k + 3}{p_k} < \frac{2k}{p_k} < \frac{2}{\log k} < \frac{1}{\sqrt{n}}.
\]

Then we get \(1 = p^m(2(1 + \varepsilon)^m - p_k^{n-m})\). So for \(n > 7\),

\[
p_k \leq p_k^{n-m} < 2(1 + \varepsilon)^m < 2\left(1 + \frac{1}{\sqrt{n}} \right)^n < 2e^{n^{1/2}} < \frac{1}{2} e^{n^{3/4}} < \frac{1}{2} p_{k+1}.
\]

which is impossible. Hence, we obtain \(S_0 \cap S_1 = \emptyset\).

Let \((p_{k+1}, p_k, m, n) \in S_0 \cap S_2\). We consider the linear form

\[
\Lambda = m \log p_{k+1} - n \log p_k + \log 2.
\]

Then we have \(\Lambda < e^{\Lambda - 1} = \frac{1}{p_k^\varepsilon}\). So \(\log \Lambda < -n \log p_k\). Now we apply Lemma 2.4 with \(D = 1\), \(\alpha_1 = p_{k+1}\), \(\alpha_2 = p_k\) and \(\alpha_3 = 2\). Therefore, we take \(A_1 = \log p_{k+1}\), \(A_2 = \log p_k\), \(A_3 = 2\), \(B = n\). So we have

\[
\log \Lambda > -9.65 \cdot 10^{10} \log p_{k+1} \log p_k \log(en).
\]

Therefore we have

\[
\frac{n}{\log p_{k+1} \log(en)} < 9.65 \cdot 10^{10}.
\]

On the other hand, from \(k + 1 < e^{n^{3/4}}\), we have for \(n > 4\),

\[
p_{k+1} < 2(k + 1) \log(k + 1) = 2e^{n^{3/4}} \cdot n^{3/4} < e^{n^{3/5}}.
\]

So we obtain

\[
n < 7 \cdot 10^{65}.
\]

Then by Lemma 2.7, we have \(S_0 \cap S_2\) is finite.

This complete the proof of Theorem 1.3.

**References**

On a Diophantine equation


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