Join Reductions and Join Saturation Reductions of Abstract Knowledge Bases

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Abstract

The covering approximation spaces are an improvement of traditional rough set model to deal with more complex practical problems. This paper generalizes covering approximation spaces to abstract knowledge bases. The concepts of join reductions, join saturations and join saturation reductions of abstract knowledge bases are introduced. Global properties of join saturations and join saturation reductions are investigated. It is proved that for an abstract knowledge base on a finite universe, there is a unique join saturation reduction. Some sufficient and necessary conditions for a join reduction being a join saturation reduction of a finite abstract knowledge base are obtained.

Keywords: sup-semilattice; poset; abstract knowledge base; covering approximation space; join (saturation) reduction

1 Introduction

Rough set theory [4, 9] is an important tool for dealing with fuzzyness and uncertainty of knowledge, and has become an active branch of information sciences. In Pawlak's original rough set theory, partition or equivalent relation is

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restrictive for many applications. To solve this problem, many scholars have extended the equivalent relation to general binary relations [1, 7, 8, 9]. And using coverings of a universe to establish covering approximation space theory [9] and more researches have been done [1, 2, 5].

Basic opinion in rough set theory is that the knowledge (human intelligence) is the ability to classify elements which leads to the definition of a knowledge base. In [6], Xu and Zhao generalized knowledge bases to abstract knowledge bases (in short AKB). Specifically, given a universe \( U \) and a family \( \mathcal{P} \) of subsets of \( U \), then the pair \((U, \mathcal{P})\) or \( \mathcal{P} \) is called an abstract knowledge base, briefly, AKB. On one hand, viewing a generalized approximation space \((U, R)\) as the abstract knowledge base \((U \times U, R)\), we see that the concept of abstract knowledge bases is a generalization of generalized approximation spaces. In this view, to consider (saturation) reductions of abstract knowledge bases, Xu and Zhao in [6] naturally used the meet operations in the manner of ones used for generalized approximation spaces based on relations. On the other hand, a covering approximation space \((U, \mathcal{C})\), where \( \mathcal{C} \) is a covering of \( U \) is a special abstract knowledge base. So, the concept of abstract knowledge bases is also a generalization of covering approximation spaces. And it is natural to consider some kind of (saturation) reductions of abstract knowledge bases using the join operations in the manner of ones used for covering approximation spaces based on coverings. So, this paper will introduce the concept of join reductions of abstract knowledge bases, which is similar to reductions of covering approximation spaces in [9]. Dual to saturations and saturation reductions in [6], the concepts of join saturations and join saturation reductions are also introduced in this paper. It is proved that for an abstract knowledge base on finite universe, there is a unique join saturation reduction. Sufficient and necessary conditions for a join reduction being a join saturation reduction of a finite abstract knowledge base are obtained.

2 Preliminaries

We give some basic concepts used in the sequel. Other unstated concepts please refer to [1, 3, 6, 9].

Definition 2.1. [6] Let \( \mathcal{P} \neq \emptyset \) be a family of subsets of \( U \). Then \((U, \mathcal{P})\) is called an abstract knowledge base, or, briefly, an AKB. Sometimes we also call \( \mathcal{P} \) an abstract knowledge base, or, briefly, an AKB.

Definition 2.2. [1] Let \( U \) be a nonempty set, \( \mathcal{C} \) a family of subsets of \( U \). If none subsets in \( \mathcal{C} \) is empty, and \( \cup \mathcal{C} = U \), then the ordered pair \((U, \mathcal{C})\) is called a covering approximation space. The family \( \mathcal{C} \) is called a covering of \( U \).
Definition 2.3. [3] Let $L$ be a poset. If every pair of elements in $L$ has a sup, then $L$ is called an \textit{sup-semilattice}. A subset $X \subseteq L$ is said to be \textit{directed} if every pair of elements $a, b \in X$, there is $c \in X$ such that $c \geq a$ and $c \geq b$.

Remark 2.4. Every sup-semilattice itself is directed.

Definition 2.5. [9] Let $C$ be a covering of a universe $U$ and $A \in C$. If $A$ is a union of some sets in $C \setminus \{A\}$, we say $A$ is \textit{reducible}, otherwise $A$ is \textit{irreducible}. If every element of $C$ is an irreducible element, we say that $C$ is \textit{irreducible}, otherwise $C$ is \textit{reducible}.

Definition 2.6. [9] Let $C$ be a covering of a universe $U$, $\mathcal{D} \subseteq C$, if $\mathcal{D}$ is irreducible and $\bigcup \mathcal{D} = \bigcup C$, then $\mathcal{D}$ is called a \textit{reduction} of $C$.

3 Join (saturation) reductions of AKBs

In this section, we introduce join reductions of an abstract knowledge base (AKB). Dual to saturations and saturation reductions of AKB in [6], the concepts of join saturations and join saturation reductions will be introduced and their properties will be explored.

Definition 3.1. Let $\mathcal{P}$ be an abstract knowledge base on $U$, $\mathcal{Q} \subseteq \mathcal{P}$. If $\bigcup \mathcal{Q} = \bigcup \mathcal{P}$ and $\forall R \in \mathcal{Q}, \bigcup \mathcal{Q} \neq \bigcup (\mathcal{Q} \setminus \{R\})$, then $\mathcal{Q}$ is called a \textit{join reduction} of $\mathcal{P}$.

Remark 3.2. Let $\mathcal{P}$ be an AKB, when $\bigcup \mathcal{P} = U$, by Definitions 2.2 and 2.6, $(U, \mathcal{P})$ is a covering approximation space and the join reductions of the AKB $\mathcal{P}$ are the reductions of the covering approximation space $(U, \mathcal{P})$.

Definition 3.3. Let $\mathcal{P}$ be an AKB and $\mathcal{P}^\sharp$ consist of all the nonempty finite unions of $\mathcal{P}$. Then $\mathcal{P}^\sharp$ is called the \textit{join saturation} of $\mathcal{P}$. If $\mathcal{P} = \mathcal{P}^\sharp$, then $\mathcal{P}$ is called \textit{join saturated}.

Notice that generally $\mathcal{P} \subseteq \mathcal{P}^\sharp$. However, even if $(\mathcal{P}, \subseteq)$ is a sup-semilattice, one can easily construct examples with $\mathcal{P} \neq \mathcal{P}^\sharp$. If an AKB is a sup-semilattice, then itself must be directed and $\bigcup \mathcal{P} = \bigcup \mathcal{P}^\sharp$.

Definition 3.4. Let $\mathcal{P}$ be an AKB. If $\forall C \in \mathcal{P}, (\mathcal{P} \setminus \{C\})^\sharp \neq \mathcal{P}^\sharp$, then $\mathcal{P}$ is said to be \textit{minimally join saturated}.

By Definition 3.4, it is easy to show the following proposition.

Proposition 3.5. If no element $A \in \mathcal{P}$ can be expressed as a finite union of elements in $\mathcal{P} \setminus \{A\}$, then $\mathcal{P}$ is minimally join saturated.
Definition 3.6. Let $\mathcal{P}$ be an AKB, $\mathcal{P}_0 \subseteq \mathcal{P}$. If $\mathcal{P}_0$ is minimally join saturated and $\mathcal{P}_0^\sharp = \mathcal{P}^\sharp$, then $\mathcal{P}_0$ is called a join saturation reduction of $\mathcal{P}$.

It is clear that $\mathcal{P}$ is minimally join saturated iff $\mathcal{P}$ is a join saturation reduction of $\mathcal{P}$. By the minimality of join saturation reduction and Definition 3.4, the following proposition is obvious and its proof is omitted.

Proposition 3.7. Let $\mathcal{P}_0, \mathcal{P}_1$ be join saturation reductions of $\mathcal{P}$ and $\mathcal{P}_0 \subseteq \mathcal{P}_1$, then $\mathcal{P}_0 = \mathcal{P}_1$.

Proposition 3.8. Let $\mathcal{P}$ and $\mathcal{P}'$ be AKBs, $\mathcal{P}_0$ a join saturation reduction of $\mathcal{P}$ and $\text{min}(\mathcal{P})$ the set of all minimal elements of $\mathcal{P}$. Then $\mathcal{P}_0 \subseteq \mathcal{P}' \subseteq \mathcal{P}^\sharp$, then $\mathcal{P}_0$ is a join saturation reduction of both $\mathcal{P}'$ and $\mathcal{P}^\sharp$.

Proof. Suppose there exists $A \in \text{min}(\mathcal{P})$, $A \notin \mathcal{P}_0$. By the minimality we have $A \notin \mathcal{P}_0^\sharp$. Since $\mathcal{P}_0$ is a join saturation reduction of $\mathcal{P}$ we have $\mathcal{P}_0^\sharp = \mathcal{P}^\sharp$. Then $A \notin \mathcal{P}^\sharp$, a contradiction! So, $\text{min}(\mathcal{P}) \subseteq \mathcal{P}_0$. If $\mathcal{P}_0 \subseteq \mathcal{P}' \subseteq \mathcal{P}^\sharp$, then $\mathcal{P}_0^\sharp \subseteq (\mathcal{P}')^\sharp \subseteq (\mathcal{P}^\sharp)^\sharp = \mathcal{P}^\sharp = \mathcal{P}_0^\sharp$. So $\mathcal{P}_0^\sharp = \mathcal{P}^\sharp = (\mathcal{P}^\sharp)^\sharp$. As $\mathcal{P}_0$ is minimally join saturated, we have $\mathcal{P}_0$ is a join saturation reduction of both $\mathcal{P}'$ and $\mathcal{P}^\sharp$. □

Proposition 3.9. Let $\mathcal{P}$ be an AKB, $\mathcal{P}_0$ a join saturation reduction of $\mathcal{P}$. Then for every $C \in \mathcal{P}_0$, there are no finite elements $K_1, \ldots, K_m \in \mathcal{P}_0 \setminus \{C\}$ such that $C = \bigcup_{i=1}^m K_i$.

Proof. Suppose there exists finite elements $K_1, \ldots, K_m \in \mathcal{P}_0 \setminus \{C\}$ such that $C = \bigcup_{i=1}^m K_i$. Then $C \in (\mathcal{P}_0 \setminus \{C\})^\sharp$ and $(\mathcal{P}_0 \setminus \{C\})^\sharp \subseteq \mathcal{P}_0$. So $\mathcal{P}^\sharp = \mathcal{P}_0^\sharp \subseteq (\mathcal{P}_0 \setminus \{C\})^\sharp$ which gives that $\mathcal{P}^\sharp = \mathcal{P}_0^\sharp = (\mathcal{P}_0 \setminus \{C\})^\sharp$, contradicting to the assumption that $\mathcal{P}_0$ is a join saturation reduction of $\mathcal{P}$. So there are no finite elements $K_1, \ldots, K_m \in \mathcal{P}_0 \setminus \{C\}$ such that $C = \bigcup_{i=1}^m K_i$. □

When the universe $U$ is finite, an AKB $\mathcal{P}$ on $U$ in the reverse order of set-inclusion is a finite poset. By the Zorn’s Lemma, we know that $\text{min}(\mathcal{P}) \neq \emptyset$. With this observation, we can prove the following theorem which reveals the existence of join saturation reduction.

Theorem 3.10. If $\mathcal{P}$ is an abstract knowledge base on a finite universe $U$, then $\mathcal{P}$ has at least a join saturation reduction.

Proof. Inductively by the above observation we construct a series $K_0, K_1, \ldots, K_n \cdots$ of subsets of $\mathcal{P}$ such that $K_0 = \text{min}(\mathcal{P}) \subseteq \mathcal{P}$, $K_1 = \text{min}(\mathcal{P} \setminus K_0^\sharp) \subseteq \mathcal{P}$, $K_2 = \text{min}(\mathcal{P} \setminus (K_0 \cup K_1)^\sharp) \subseteq \mathcal{P}$, $\cdots$, $K_n = \text{min}(\mathcal{P} \setminus (K_0 \cup \cdots \cup K_{n-1})^\sharp)$, $\ldots$

Noticing that $\text{min}(\mathcal{P}) \neq \emptyset$, we know that the family $\mathcal{P} \setminus (K_0 \cup \cdots \cup K_i)^\sharp$ is strictly decreased. Since $U$ and $\mathcal{P}$ are finite, there is some $n$ such that $\mathcal{P} \setminus (K_0 \cup \cdots \cup K_n)^\sharp = \emptyset$, and thus $K_i = \emptyset$ for all $i \geq n + 1$. Set $\mathcal{P}_0 = \bigcup_{i=0}^n K_i \subseteq \mathcal{P}$. We have that $\mathcal{P}_0^\sharp = (\bigcup_{i=0}^n K_i)^\sharp \subseteq \mathcal{P}^\sharp$ and $(\bigcup_{i=0}^n K_i)^\sharp \supseteq \mathcal{P}$, giving that
For each $C \in P_0$, there exists $i_0 \leq n$ such that $C \in K_{i_0}$. We assert that $C \not\in (P_0 \setminus \{C\})^2$. In fact, if $C \in (P_0 \setminus \{C\})^2 \subseteq P^2$, then there is a nonempty finite set $R \subseteq \bigcup_{i=0}^{n} K_i \setminus \{C\}$ such that $C = \bigcup \{R| R \in R\}$. Set $R_1 = R \cap \bigcup_{i=0}^{n} K_i, R_2 = R \cap \bigcup_{i=i_0}^{n} K_i$. Then $R_1 \cup R_2 = R$ and $C = (\cup R_1) \cup (\cup R_2)$. Notice that $R_2 \subseteq \bigcup_{i=i_0}^{n} K_i \subseteq P \setminus (\bigcup_{i=0}^{n-1} K_i)^2$. If $R_2 \neq \emptyset$, then for each $R \in R_2$, it follows from $R \neq C \in K_{i_0}$ (i.e. $C$ is a minimal element in $P \setminus (\bigcup_{i=0}^{n-1} K_i)^2$) and $K_i \cap K_j = \emptyset (\forall i \neq j)$ that $R \not\subseteq C$, a contradiction. So $R_2 = \emptyset$. Then $C = \cup R_1 \not\in K_{i_0}$, contradicting to $C \in K_{i_0}$. Then the assertion is proved. By the assertion, we have $(P_0 \setminus \{C\})^2 \neq P^2$. By Definition 3.6, we see that $P_0$ is a join saturation reduction of $P$.

**Theorem 3.11.** Let $P$ be an AKB on a finite universe $U$, $P_0$ the join saturation reduction constructed in Theorem 3.10 and $P_1$ another join saturation reduction of $P$. Then $P_0 \subseteq P_1$ and $P_0 = P_1$ is the unique join saturation reduction of $P$.

**Proof.** We need to prove $K_i \subseteq P_1$ ($i = 0, \cdots , n$). To this end, we use the mathematical induction.

(1) For $K_0 = \mathit{min}(P)$ and $\forall C \in K_0$ there exists $A_s \in P_1 (s = 0, \cdots, m)$ such that $C = \cup A_s$. Since $C$ is a minimal element, there exists $s_0$ such that $C = A_{s_0} \in P_1$. So, $K_0 \subseteq P_1$.

(2) Assume that when $i \leq j$, $K_i \subseteq P_1$. Then

(3) We show $K_{j+1} \subseteq P_1$. In fact, $\bigcup_{i=0}^{j} K_i \subseteq P_1$ and $P_1 \setminus (\bigcup_{i=0}^{j} K_i)^2 \subseteq P \setminus (\bigcup_{i=0}^{j} K_i)^2 \subseteq \uparrow K_{j+1}$. Thus, for each $R \in K_{j+1}$ we have that $R \not\in (\bigcup_{i=0}^{j} K_i)^2$. Since $P_1$ is a join saturation reduction, there is a finite family $R = R_1 \cup R_2 \subseteq P_1$ such that $R = (\cup R_1) \cup (\cup R_2)$, where $R_1 \subseteq P_1 \setminus (\bigcup_{i=0}^{j} K_i)^2$ and $R_2 \subseteq P_1 \cap (\bigcup_{i=0}^{j} K_i)^2$. By $R_1 \subseteq \uparrow K_{j+1}$ and that $R$ is a minimal element of $\uparrow K_{j+1}$, we have that $R_1 = \emptyset$ or $R_1 = \{R\}$. If $R_1 = \emptyset$, then $R = \cup R_2 \in (\bigcup_{i=0}^{j} K_i)^2$, contradicting to $R \not\in (\bigcup_{i=0}^{j} K_i)^2$. So, $R_1 = \{R\}$ and $R \in R_1 \subseteq P_1$. So, $K_{j+1} \subseteq P_1$.

By the principle of mathematical induction, we have that $K_i \subseteq P_1$ ($i = 0, \cdots , n$) and $P_0 \subseteq P_1$. Since $P_0$ and $P_1$ are both minimal join saturated, by Proposition 3.7, $P_0 = P_1$.

**4 Relationships between join reductions and join saturation reductions**

In this section we explore relationships between join reductions and the join saturation reduction of an AKB.
**Theorem 4.1.** Let $\mathcal{P}$ be an AKB and $\mathcal{P}_0 \subseteq \mathcal{P}$. Then $\mathcal{P}_0$ is both the join saturation reduction and a join reduction of $\mathcal{P}$ if and only if $\mathcal{P}_0 = \min(\mathcal{P})$, $\mathcal{P}^\sharp_0 = \mathcal{P}^\sharp$ and $\forall C \in \mathcal{P}_0, \cup(\mathcal{P}_0 \setminus \{C\}) \neq \cup \mathcal{P}$.

**Proof.** $\Leftarrow$: It follows from Definitions 3.1 and 3.6, and Theorems 3.10 and 3.11.

$\Rightarrow$: Since $\mathcal{P}_0$ is a join saturation reduction, we have $\mathcal{P}^\sharp_0 = \mathcal{P}^\sharp$. Since $\mathcal{P}_0$ is also a join reduction, we have $\forall C \in \mathcal{P}_0$, $\cup(\mathcal{P}_0 \setminus \{C\}) \neq \cup \mathcal{P}$ = $\cup \mathcal{P}_0$. To show $\mathcal{P}_0 = \min(\mathcal{P})$, by Theorems 3.10 and 3.11, we have $\mathcal{P}_0 = K_0 \cup K_1 \cup \cdots \cup K_n$, where $K_0 = \min(\mathcal{P})$ and $K_i = \min(\mathcal{P} \setminus (K_0 \cup K_1 \cup \cdots \cup K_{i-1})^\sharp)$ ($i = 1, 2, \cdots, n$).

Let $C^\prime \in \bigcup_{i=1}^n K_i$. Then there is $C_\xi \in K_0 \subseteq \mathcal{P}_0$ such that $C_\xi \subseteq C^\prime$. Notice that $C_\xi$ is a minimal element, we have $\cup(\mathcal{P}_0 \setminus \{C_\xi\}) = \cup \mathcal{P}_0 = \cup \mathcal{P}$, contradicting to $\forall C \in \mathcal{P}_0$, $\cup(\mathcal{P}_0 \setminus \{C\}) \neq \cup \mathcal{P} = \cup \mathcal{P}_0$, showing that $\bigcup_{i=1}^n K_i = \emptyset$ and $\mathcal{P}_0 = K_0 = \min(\mathcal{P})$. $\square$

**Remark 4.2.** (1) By the above theorem, the join saturation reduction of an AKB $\mathcal{P}$ may not be a join reduction of $\mathcal{P}$. And a join reduction of an AKB $\mathcal{P}$ may not be the join saturation reduction of $\mathcal{P}$.

(2) Even if an AKB has a join reduction which is also the join saturation reduction, then the AKB may also has other join reductions.

**Example 4.3.** Let $U = \{a, b, c\}$, $\mathcal{P}_1 = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, $\mathcal{P}_2 = \{\{a\}, \{b\}, \{c\}, \{a, b\}\}$. For $\mathcal{P}_1$, the join reductions are $\{\{a\}, \{b\}\}$ and $\{\{a, b\}, \{b, c\}\}$; the join saturation reduction is $\{\{a\}, \{b\}, \{b, c\}\}$, revealing that there is no join reduction which is also the join saturation reduction of $\mathcal{P}_1$. It is easy to check that $\{\{a\}, \{b\}, \{c\}\}$ is both a join reduction and the join saturation reduction of $\mathcal{P}_2$. However, $\mathcal{P}_2$ has also another join reduction $\{\{c\}, \{a, b\}\}$.

**References**


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