Symmetries in Lightlike Hypersurfaces of Indefinite Kenmotsu Manifolds

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Abstract

In this paper, we deal with the symmetric property of lightlike hypersurfaces of indefinite Kenmotsu manifolds tangent to the vector field. We prove that there exist no weakly Ricci $\eta$-Einstein (or screen locally conformal) lightlike hypersurfaces in indefinite Kenmotsu manifolds, tangent to the structure vector field, if $\alpha + \beta + \gamma$ is nowhere zero, where $\alpha$, $\beta$ and $\gamma$ are 1-form defined on the submanifold. We also prove that the geometry of the special weakly lightlike hypersurfaces is closely related to that of geodesibility and umbilicality of its tangent space and screen distribution, respectively. Under some conditions, a special weakly Ricci symmetric screen locally (or globally) conformal (or $\eta$-Einstein or Einstein) lightlike hypersurface of an indefinite Kenmotsu space form, tangent to the structure vector field, is locally symmetric, semi-symmetric and Ricci semi-symmetric.

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1 Introduction

The general theory of lightlike submanifolds was introduced and presented by Duggal and Bejancu. They, Duggal and Bejancu in [1] introduced a non-degenerate screen distribution to construct a non-intersecting lightlike transversal vector bundle of the tangent bundle. The induced objects on a lightlike submanifold depend on the choice of a screen distribution which, in general, is not unique. Several authors have studied lightlike hypersurfaces of indefinite almost contact manifolds (see [7], [8], [9], [10] and many more references therein). In this paper, we study symmetries in lightlike hypersurfaces of indefinite Kenmotsu manifolds $\overline{M}$, tangent to the structure vector field, by introducing the condition of screen conformality on the Ricci tensor. We then pay, in Sections 3 and 4, a specific attention to Ricci symmetries (Weakly Ricci symmetry, Special weakly Ricci symmetry and Ricci semi-symmetry) and locally symmetry. A relationship between symmetries is established.

2 Preliminaries

Let $\overline{M}$ be a $(2n + 1)$-dimensional manifold endowed with an almost contact structure $(\bar{\phi}, \xi, \eta)$, i.e. $\bar{\phi}$ is a tensor field of type $(1,1)$, $\xi$ is a vector field, and $\eta$ is a 1-form satisfying

$$\bar{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \bar{\phi} = 0 \text{ and } \bar{\phi} \xi = 0. \quad (1)$$

Then $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\bar{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\bar{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\bar{X}, \bar{Y}$ on $\overline{M}$ [5]

$$\eta(\bar{X}) = \bar{g}(\xi, \bar{X}), \quad \bar{g}(\bar{\phi} \bar{X}, \bar{\phi} \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X}) \eta(\bar{Y}). \quad (2)$$

If, moreover,

$$(\nabla_{\bar{X}} \bar{\phi}) \bar{Y} = \bar{g}(\bar{\phi} \bar{X}, \bar{Y}) \xi - \eta(\bar{Y}) \bar{\phi} \bar{X}, \quad (3)$$

where $\nabla$ is the Levi-Civita connection for the semi-Riemannian metric $\bar{g}$, we call $\bar{M}$ an indefinite Kenmotsu manifold. Here, without loss of generality, the vector field $\xi$ is assumed to be spacelike, that is, $\bar{g}(\xi, \xi) = 1$.

In this case, the relation (3) implies

$$\nabla_{\bar{X}} \xi = \bar{X} - \eta(\bar{X}) \xi. \quad (4)$$

A plane section $\sigma$ in $T_p \overline{M}$ is called a $\bar{\phi}$-section if it is spanned by $\bar{X}$ and $\bar{\phi} \bar{X}$, where $\bar{X}$ is a unit tangent vector field orthogonal to $\xi$. The sectional curvature of a $\bar{\phi}$-section $\sigma$ is called a $\bar{\phi}$-sectional curvature. If a Kenmotsu manifold $\overline{M}$
has constant $\bar{\phi}$-sectional curvature $c$, then, by virtue of the Proposition 12 in [12], the curvature tensor $\bar{R}$ of $\bar{M}$ is given by, for any $X, Y, Z \in \Gamma(TM),
\begin{align*}
\bar{R}(X, Y)Z &= \frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\
&\quad - \eta(Y)\eta(Z)X + \bar{g}(X, Z)\eta(Y)\xi - \bar{g}(Y, Z)\eta(X)\xi \\
&\quad + \bar{g}(\bar{\phi} Y, Z)\bar{\phi} X - \bar{g}(\bar{\phi} X, Z)\bar{\phi} Y - 2\bar{g}(\bar{\phi} X, Y)\bar{\phi} Z\}. \tag{5}
\end{align*}

A Kenmotsu manifold $\bar{M}$ of constant $\bar{\phi}$-sectional curvature $c$ will be called Kenmotsu space form and denoted by $\bar{M}(c)$.

Let $(\bar{M}, \bar{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index $s$, $0 < s < 2n + 1$ and let $(M, g)$ be a hypersurface of $\bar{M}$, with $g = \bar{g}|_M$. $M$ is said to be a lightlike hypersurface of $\bar{M}$ if $g$ is of constant rank $2n - 1$ and the orthogonal complement $TM^\perp$ of tangent space $TM$, defined as

$$TM^\perp = \bigcup_{p \in M} \{ Y_p \in T_p \bar{M} : \bar{g}_p(X_p, Y_p) = 0, \forall X_p \in T_p M \}, \tag{6}$$

is a distribution of rank 1 on $M$ [1]: $TM^\perp \subset TM$ and then coincides with the radical distribution $\text{Rad} TM = TM \cap TM^\perp$. A complementary bundle of $TM^\perp$ in $TM$ is a rank $2n - 1$ non-degenerate distribution over $M$. It is called a screen distribution and is often denoted by $S(TM)$. Existence of $S(TM)$ is secured provided $M$ is paracompact. However, in general, $S(TM)$ is not canonical (thus it is not unique) and the lightlike geometry depends on its choice.

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface [1].

**Theorem 2.1. (Duggal-Bejancu)** Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over $M$ such that for any non-zero section $E$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique section $N$ of $N(TM)$ on $U$ satisfying

$$\bar{g}(N, E) = 1 \text{ and } \bar{g}(N, N) = \bar{g}(N, W) = 0, \forall W \in \Gamma(S(TM)|_U). \tag{7}$$

Throughout the project, all manifolds are supposed to be paracompact and smooth.

**Theorem 2.2.** Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\bar{M}(c)$, with $\xi \in TM$. Then the Ricci tensor $\text{Ric}$ of $M$ is given by, for any $X, Y \in \Gamma(TM),

$$\text{Ric}(X, Y) = ag(X, Y) + B(X, Y)\text{tr} A_N - B(A_N X, Y), \tag{8}$$
where $a = -(2n - 1)$ and trace $tr$ is written with respect to $g$ restricted to $S(TM)$.

Note that the Ricci tensor does not depend on the choice of the vector field $E$ of the distribution $TM^\perp$. We also state the following theorem

**Theorem 2.3.** [9] Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. If the second fundamental form $h$ of $M$ is parallel, then $M$ is totally geodesic.

**Lemma 2.4.** Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Then, the covariant derivative of the Ricci tensor $Ric$ of $M$ is given by, for any $X, Y, Z \in \Gamma(TM)$,

$$(\nabla_XRic)(Y,Z) = a\{B(X,Y)\theta(Z) + B(X,Z)\theta(Y)\} + \nabla_XB(Y,Z)trA_N$$

$$- \{\nabla_XB(A_NY,Z) + B((\nabla_XA_N)Y,Z)\} + B(Y,Z)X \cdot trA_N.$$  (9)

## 3 Weakly and special weakly Ricci symmetric lightlike hypersurfaces

Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$.

**Definition 3.1.** A submanifold $M$ of a semi-Riemannian manifold $\overline{M}$ is said to be $\eta$-Einstein if its Ricci tensor $Ric$ satisfies

$$Ric(X,Y) = k_1g(X,Y) + k_2\eta(X)\eta(Y),$$  (10)

where the non-zero functions $k_1$ and $k_2$ are not necessarily constant on $M$. If $k_2 = 0$, then $M$ is said to be Einstein.

By Theorem 2.2, the Ricci tensor of $M$ is given by, for any $X, Y \in \Gamma(TM)$,

$$Ric(X,Y) = ag(X,Y) + B(X,Y)trA_N - B(A_NX,Y),$$  (11)

where $a = -(2n - 1)$ and trace $tr$ is written with respect to $g$ restricted to $S(TM)$. From this relation, we have

$$Ric(X,Y) - Ric(Y,X) = B(A_NX,Y) - B(A_NY,X).$$  (12)

This means that the Ricci tensor of a lightlike hypersurface $M$ of an indefinite Kenmotsu space form $\overline{M}(c)$ is not symmetric in general.

We recall the definition of screen conformal lightlike hypersurfaces of a semi-Riemannian manifold $\overline{M}$.  

Definition 3.2. A lightlike hypersurface \((M, g, S(TM))\) of a semi-Riemannian manifold is screen locally conformal if the shape operator \(A_N\) and \(A_E\) of \(M\) and its screen distribution \(S(TM)\), respectively, are related by
\[
A_N = \varphi A_E^*\quad \tag{13}
\]
where \(\varphi\) is a non-vanishing smooth function on \(\mathcal{U}\) in \(M\). In case \(\mathcal{U} = M\) the screen conformality is said to be global.

Theorem 3.3. Let \((M, g, S(TM))\) be a locally (or globally) screen conformal lightlike hypersurface of an indefinite Kenmotsu space form \(\overline{M}(c)\), with \(\xi \in TM\). Then the Ricci tensor of the induced connection \(\nabla\) is symmetric.

A submanifold \(M\) is said to be weakly Ricci symmetric if there exist 1-forms \(\alpha, \beta\) and \(\gamma\) such that the condition [4]:
\[
(\nabla_X Ric)(Y, Z) = \alpha(X)Ric(Y, Z) + \beta(Y)Ric(X, Z) + \gamma(Z)Ric(Y, X),
\]
holds for any vector fields \(X, Y, Z \in \Gamma(TM)\). We denote this kind of \(2n\)-dimensional submanifold by \((WRS)_{2n}\).

Suppose that \(M\) is an \(\eta\)-Einstein lightlike hypersurface, then the Ricci tensor \(Ric\) of \(M\) satisfies
\[
Ric(Y, Z) = k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z),\quad \tag{15}
\]
where non zero functions \(k_1\) and \(k_2\) are not necessarily constant on \(M\) and its covariant derivative is given by
\[
\nabla_X Ric(Y, Z) = \nabla_X k_1 g(Y, Z) + k_1 \{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\}
+ \nabla_X k_2 \eta(Y)\eta(Z)\} + k_2 \{\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z\}.\quad \tag{16}
\]
We recall, using (8), that
\[
Ric(\xi, \xi) = a = -(2n - 1).\quad \tag{17}
\]
Therefore, for an \(\eta\)-Einstein lightlike hypersurface we get \(Ric(\xi, \xi) = k_1 + k_2\), that is, \(k_1 + k_2 = -(2n - 1)\). This implies that the covariant derivatives of \(k_1\) and \(k_2\) are opposite, that is, for any \(X \in \Gamma(TM)\),
\[
\nabla_X k_1 = -\nabla_X k_2.\quad \tag{18}
\]
If, moreover \(M\) is weakly Ricci symmetric, we have
\[
\nabla_X Ric(Y, Z) = \alpha(X)\{k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z)\} + \beta(Y)\{k_1 g(X, Z)
+k_2 \eta(X)\eta(Z)\} + \gamma(Z)\{k_1 g(Y, X) + k_2 \eta(Y)\eta(X)\}.\quad \tag{19}
\]
Theorem 3.4. Let $M$ be a weakly Ricci symmetric $\eta$-Einstein lightlike hypersurface of an indefinite Kenmotsu space form $(\mathbb{M}^{2n+1}(c)) (n>1)$ with $\xi \in TM$. Then, the 1-forms $\alpha$, $\beta$ and $\gamma$ satisfy $\alpha + \beta + \gamma = 0$.

Corollary 3.5. There exist no weakly Ricci-symmetric $\eta$-Einstein lightlike hypersurface of an indefinite Kenmotsu space form $(\mathbb{M}^{2n+1}(c)) (n>1)$ with $\xi \in TM$ if $\alpha + \beta + \gamma$ is not everywhere zero.

The induced Ricci tensor $Ric$ of submanifold $M$ is said to be parallel if, for any $X, Y, Z \in \Gamma(TM)$,

$$\nabla_X Ric(Y, Z) = 0. \tag{21}$$

Theorem 3.6. Let $M$ be a weakly Ricci symmetric $\eta$-Einstein lightlike hypersurface of an indefinite Kenmotsu manifold $\mathbb{M}^{2n+1}$ for $n>1$ with $\xi \in TM$ such that the Ricci tensor $Ric$ of $M$ is parallel. Then the 1-forms $\alpha$, $\beta$ and $\gamma$ satisfy $\alpha = 0$ and $\beta + \gamma = 0$.

Proof. Let $M$ be a weakly Ricci symmetric $\eta$-Einstein lightlike hypersurface of an indefinite Kenmotsu manifold $\mathbb{M}^{2n+1}$ for $n>1$ with $\xi \in TM$. By Theorem 3.4, we have, for any $X \in \Gamma(M)$, $\alpha(X) + \beta(X) + \gamma(X) = 0$, where $\alpha(X) = \alpha(\xi)\eta(X)$, $\beta(X) = -(\alpha(\xi) + \gamma(\xi))\eta(X)$, $\gamma(X) = \gamma(\xi)\eta(X)$ and $\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0$. If the Ricci tensor $Ric$ of $M$ is parallel, using (19) we have

$$0 = \alpha(X)\{k_1g(Y, Z) + k_2\eta(Y)\eta(Z)\} + \beta(Y)\{k_1g(Y, Z) + k_2\eta(Y)\eta(Z)\} + \gamma(Z)\{k_1g(Y, X) + k_2\eta(Y)\eta(X)\}, \tag{22}$$

which leads, by taking $Y = V$ and $Z = U$, to $k_1\alpha(\xi) = 0$, i.e. $\alpha(\xi) = 0$. Therefore $\alpha(X) = 0$ and $\beta(X) + \gamma(X) = 0$. This completes the proof. $\blacksquare$

It is easy to show that the above theorem still holds for $k_2 = 0$, i.e. for a weakly Ricci symmetric Einstein lightlike hypersurface.
**Definition 3.7.** A non zero Ricci tensor of a lightlike hypersurface $M$ is said to be cyclic parallel if, for any $X, Y, Z \in \Gamma(TM)$,

$$
\mathcal{C}(\nabla_X \text{Ric})(Y, Z) = 0,
$$

where $\mathcal{C}(\nabla_X \text{Ric})(Y, Z) = (\nabla_X \text{Ric})(Y, Z) + (\nabla_Y \text{Ric})(Z, X) + (\nabla_Z \text{Ric})(X, Y)$.

Let $M$ be a weakly lightlike hypersurface of an indefinite Kenmotsu space form with $\xi \in TM$ such its Ricci tensor $\text{Ric}$ of $M$ is symmetric. We then have, for $X, Y, Z \in \Gamma(TM)$,

$$
\mathcal{C}(\nabla_X \text{Ric})(Y, Z) = \{\alpha(X) + \beta(X) + \gamma(X)\} \text{Ric}(Y, Z) \\
+ \{\alpha(Y) + \beta(Y) + \gamma(Y)\} \text{Ric}(X, Z) \\
+ \{\alpha(Z) + \beta(Z) + \gamma(Z)\} \text{Ric}(Y, X).
$$

Taking $Z = \xi$ in (24) and using the identity $\text{Ric}(\cdot, \xi) = a\eta(\cdot)$ and $\text{Ric}(\xi, \cdot) = a\eta(\cdot) - B(A\xi, \cdot)$, one obtains

$$
\mathcal{C}(\nabla_X \text{Ric})(Y, \xi) = a\{\alpha(X) + \beta(X) + \gamma(X)\}\eta(Y) \\
+ a\{\alpha(Y) + \beta(Y) + \gamma(Y)\}\eta(X) \\
+ \{\alpha(\xi) + \beta(\xi) + \gamma(\xi)\} \text{Ric}(Y, X). 
$$

For $Y = \xi$, this relation becomes

$$
\mathcal{C}(\nabla_X \text{Ric})(\xi, \xi) = a\{\alpha(X) + \beta(X) + \gamma(X)\} \\
+ 2a\{\alpha(\xi) + \beta(\xi) + \gamma(\xi)\}\eta(X) \\
- \{\alpha(\xi) + \beta(\xi) + \gamma(\xi)\} B(A\xi, X). 
$$

If the induced Ricci tensor $\text{Ric}$ of the lightlike hypersurface $M$ is cyclic parallel, then, using (26) and taking $X = \xi$, we have

$$
3a\{\alpha(\xi) + \beta(\xi) + \gamma(\xi)\} = 0.
$$

If $n > 1$, then $\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0$. Replacing this into (26), one obtains, for any $X \in \Gamma(TM)$, $\alpha(X) + \beta(X) + \gamma(X) = 0$. Therefore, we have the following result.

**Theorem 3.8.** There exist no weakly Ricci symmetric screen locally conformal lightlike hypersurface $M$ of an indefinite Kenmotsu space form $\text{M}^{2n+1}(c)(n > 1)$ with $\xi \in TM$ and cyclic parallel Ricci tensor if $\alpha + \beta + \gamma$ is not zero everywhere.

By Theorem 3.4 and using the relation (24), we have the following result.
Theorem 3.9. If $M$ is weakly Ricci symmetric $\eta$-Einstein (or Einstein) lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^{2n+1}$ $(n > 1)$ with $\xi \in TM$, then $M$ is cyclic parallel.

Definition 3.10. [13] A submanifold $M$ is said to be special weakly Ricci symmetric if, for all $X, Y, Z \in TM$,
\[\nabla_X \text{Ric}(Y, Z) = 2\alpha(X) \text{Ric}(Y, Z) + \alpha(Y) \text{Ric}(X, Z) + \alpha(Z) \text{Ric}(Y, X),\]
(28)
where $\alpha$ is a 1-form. Such a submanifold is denoted by $(\text{SWRS})_{2n}$.

Suppose that $M$ is $(\text{SWRS})_{2n}$. If the induced Ricci tensor $\text{Ric}$ of $M$ is cyclic parallel, then, using (24), we have
\[4\alpha(X) \text{Ric}(Y, Z) + 4\alpha(Y) \text{Ric}(Z, X) + 4\alpha(Z) \text{Ric}(X, Y) = 0\]
(29)
Taking $Z = \xi$ in (29), we get
\[4a\alpha(X) \eta(Y) + 4a\alpha(Y) \eta(X) - 4\alpha(Y) B(AN\xi, X) + 4\alpha(\xi) \text{Ric}(X, Y) = 0,\]
(30)
which implies, by taking $X = \xi$, that
\[8a\alpha(\xi) \eta(Y) + 4a\alpha(Y) = 0,\]
(31)
and for $Y = \xi$, we have $12a\alpha(\xi) = 0$, that is, $\alpha(\xi) = 0$.

Again, taking $X = E$, $Y = \xi$ and $Z = \xi$ in (29), we obtain $\alpha(E) = 0$. Similarly, we have $\alpha(V) = \alpha(U) = \alpha(F_i) = 0$. Therefore,
\[\alpha(X) = 0, \quad \forall X \in \Gamma(TM).\]
(32)
We have

Theorem 3.11. There exist no special weakly Ricci symmetric screen locally conformal lightlike hypersurfaces $M$ of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)(n > 1)$ with $\xi \in TM$ and Ricci tensor cyclic parallel if the 1-form $\alpha \neq 0$.

Corollary 3.12. If $M$ is a special weakly Ricci symmetric screen locally conformal lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^{2n+1}(c)(n > 1)$ with $\xi \in TM$ and the 1-form $\alpha \neq 0$, then $M$ cannot be Einstein. If the Ricci tensor $\text{Ric}$ of $M$ is parallel, then $M$ cannot be $\eta$-Einstein.

Let $M$ be a $(\text{SWRS})_{2n}$. Then, replacing $Z$ by $\xi$ into the Definition 3.10 and since $\text{Ric}(\cdot, \xi) = a\eta(\cdot)$, we get
\[(\nabla_X \text{Ric})(Y, \xi) = 2\alpha(X) \text{Ric}(Y, \xi) + \alpha(Y) \text{Ric}(X, \xi) + \alpha(\xi) \text{Ric}(Y, X)\]
\[= 2a\alpha(X) \eta(Y) + a\alpha(Y) \eta(X) + \alpha(\xi) \text{Ric}(Y, X)\]
(33)
Putting $X = Y = \xi$ into this relation, one obtains $(\nabla_\xi \text{Ric})(\xi, \xi) = 4\alpha(\xi)$. Since $(\nabla_\xi \text{Ric})(\xi, \xi) = 0$, we have $\alpha(\xi) = 0$.

Taking $X = \xi$ in the covariant derivative given in Definition 3.10, we get

\[
(\nabla_\xi \text{Ric})(Y, Z) = 2\alpha(\xi)\text{Ric}(Y, Z) + \alpha(Y)\text{Ric}(\xi, Z) + \alpha(Z)\text{Ric}(Y, \xi)
\]

\[
= 2\alpha(\xi)\text{Ric}(Y, Z) + a\alpha(Y)\eta(Z) - \alpha(Y)B(A_N \xi, Z)
\]

\[
+ a\alpha(Z)\eta(Y).
\]

which implies, for $Y = \xi$ and $Z = U$, that

\[
(\nabla_\xi \text{Ric})(\xi, U) = 2\alpha(\xi)\text{Ric}(\xi, U) + a\alpha(\xi)\eta(U) + a\alpha(U)\eta(\xi) = a\alpha(U).
\]

On the other hand,

\[
(\nabla_\xi \text{Ric})(\xi, U) = \xi \cdot \text{Ric}(U, \xi) - \text{Ric}(\nabla_\xi U, \xi) - \text{Ric}(U, \nabla_\xi \xi)
\]

\[
= -\text{Ric}(\nabla_\xi U, \xi) = -a\eta(\nabla_\xi U) = 0.
\]

From these last two relations and since $n > 1$, one obtains $\alpha(U) = 0$. To find the general form of $\alpha(V)$ where $V = -\phi E$ we replace $X$ by $\xi$, $Y$ by $\xi$ and $Z$ by $V$ in the special weakly symmetric formula and we get

\[
(\nabla_\xi \text{Ric})(\xi, V) = 2\alpha(\xi)\text{Ric}(\xi, V) + a\alpha(\xi)\eta(V) + a\alpha(V)\eta(\xi) = a\alpha(V).
\]

Since $(\nabla_\xi \text{Ric})(\xi, V) = \xi \cdot \text{Ric}(V, \xi) - \text{Ric}(\nabla_\xi V, \xi) - \text{Ric}(V, \nabla_\xi \xi) = 0$, therefore $\alpha(V) = 0$. For $Z = \xi$ and $Y = F_i \in \Gamma(D_0)$, we have

\[
(\nabla_\xi \text{Ric})(\xi, F_i) = 2\alpha(\xi)\text{Ric}(F_i, \xi) + \alpha(F_i)\text{Ric}(\xi, \xi) + \alpha(\xi)\text{Ric}(F_i, \xi)
\]

\[
= a\alpha(F_i).
\]

Since $(\nabla_\xi \text{Ric})(\xi, F_i) = \xi \cdot \text{Ric}(F_i, \xi) - \text{Ric}(\nabla_\xi F_i, \xi) - \text{Ric}(F_i, \nabla_\xi \xi) = 0$ and $n > 1$, we have $\alpha(F_i) = 0$. We conclude that, for any $X \in \Gamma(TM)$, $\alpha(X) = 0$.

We have

**Theorem 3.13.** The Ricci tensor of a special weakly Ricci symmetric screen locally (or globally) conformal (or $\eta$-Einstein or Einstein) lightlike hypersurface of an indefinite Kenmotsu space form $M^{2n+1}(c)(n > 1)$ with $\xi \in TM$, is parallel.

Suppose that $S(TM)$ is parallel, then, the shape operator $A_N$ vanishes. The relation (9) becomes

\[
(\nabla_X \text{Ric})(Y, Z) = a\{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\}.
\]

Taking $Z = E$ in this relation, one obtains

\[
(\nabla_X \text{Ric})(Y, E) = aB(X, Y).
\]

Therefore, we have the following result.
Theorem 3.14. Let \((M, g, S(TM))\) be a special weakly Ricci symmetric screen locally (or globally) conformal (or \(\eta\)-Einstein or Einstein) lightlike hypersurface of an indefinite Kenmotsu space form \(\mathbb{M}^{2n+1}(c)(n > 1)\) with \(\xi \in TM\) such that \(S(TM)\) is parallel. Then, \(M\) is totally geodesic.

4 Locally symmetric and Ricci semi symmetric lightlike hypersurfaces

Let \((M, g)\) be a lightlike hypersurface of an indefinite Kenmotsu space form \((\mathcal{M}(c), \bar{g})\) with \(\xi \in TM\). Let us consider the pair \(\{E, N\}\) on \(U \subset M\) (Theorem 2.1). Since the sectional curvature tensor \(c = -1\), the curvature tensor \(\bar{R}\) (5) of \(\mathcal{M}\) becomes, for any \(X, Y, Z \in \Gamma(TM)\),

\[
\bar{R}(X, Y)Z = \bar{g}(X, Z)Y - \bar{g}(Y, Z)X
\]

which implies that

\[
\bar{g}(\bar{R}(X, Y)Z, E) = 0,
\]

and

\[
\bar{g}(\bar{R}(X, Y)Z, N) = \bar{g}(X, Z)\theta(Y) - \bar{g}(Y, Z)\theta(X).
\]

Definition 4.1. The semi-Riemannian manifold \((\mathcal{M}, \bar{g})\) is locally symmetric, if its curvature tensor \(\bar{R}\) satisfies the condition

\[
(\nabla_{\bar{W}} \bar{R})(Y, Z)\bar{W} = 0,
\]

for any \(\bar{X}, \bar{Y}, \bar{Z}\) and \(\bar{W}\) on \(\mathcal{M}\), where \(\nabla\) is the Levi-Civita connection with respect to the metric \(\bar{g}\).

Theorem 4.2. An indefinite Kenmotsu space form \((\mathcal{M}(c), \bar{g})\) is locally symmetric.

Proof. The covariant derivative of \(\bar{R}\) is given by, for any \(\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(TM)\),

\[
(\nabla_{\bar{W}} \bar{R})(\bar{X}, \bar{Y})\bar{Z} = \nabla_{\bar{W}} \bar{R}(\bar{X}, \bar{Y})\bar{Z} - \bar{R}(\nabla_{\bar{W}} \bar{X}, \bar{Y})\bar{Z} - \bar{R}(\bar{X}, \nabla_{\bar{W}} \bar{Y})\bar{Z} - \bar{R}(\bar{X}, \bar{Y})\nabla_{\bar{W}} \bar{Z}.
\]

Using (39), one obtains

\[
\nabla_{\bar{W}} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = \nabla_{\bar{W}}(\bar{g}(\bar{X}, Z)\bar{Y} - \bar{g}(\bar{Y}, Z)\bar{X})
\]

\[
= \bar{W} \cdot \bar{g}(\bar{X}, Z)\bar{Y} + \bar{g}(\bar{X}, Z)\nabla_{\bar{W}} \bar{Y} - \bar{W} \cdot \bar{g}(\bar{Y}, Z)\bar{X}
\]

\[
- \bar{g}(\bar{Y}, Z)\nabla_{\bar{W}} \bar{X}
\]

\[
\bar{R}(\nabla_{\bar{W}} \bar{X}, \bar{Y})\bar{Z} = \bar{g}(\nabla_{\bar{W}} \bar{X}, Z)\bar{Y} - \bar{g}(\bar{Y}, Z)\nabla_{\bar{W}} \bar{X},
\]

\[
\bar{R}(\bar{X}, \nabla_{\bar{W}} \bar{Y})\bar{Z} = \bar{g}(\bar{X}, Z)\nabla_{\bar{W}} \bar{Y} - \bar{g}(\nabla_{\bar{W}} \bar{Y}, Z)\bar{X},
\]

\[
\bar{R}(\bar{X}, \bar{Y})\nabla_{\bar{W}} \bar{Z} = \bar{g}(\bar{X}, \nabla_{\bar{W}} \bar{Z})\bar{Y} - \bar{g}(\bar{Y}, \nabla_{\bar{W}} \bar{Z})\bar{X}.
\]
Putting these pieces together into (43), we have
\[(\nabla W R)(X,Y)Z = 0.\]
This completes the proof. \(\Box\)

The curvature tensor field \(R\) is given by, for any \(X, Y, Z \in \Gamma(TM)\),
\[R(X,Y)Z = g(X,Z)Y - g(Y,Z)X + B(Y,Z)A_NX - B(X,Z)A_NY,\] (44)
and
\[(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z).\] (45)
From this definition, we have \((\nabla W R)(X,Y)Z = 0, \\forall X,Y,Z \in \Gamma(TM)\).

In the sequel, we need the following Lemma.

**Lemma 4.3.** Let \((M,g,S(TM))\) be a lightlike hypersurface of an indefinite Kenmotsu manifold \((M,g)\) with \(\xi \in TM\). Then, for any \(X, Y \in \Gamma(TM)\) and \(Z \in \Gamma(S(TM))\),
\[g((\nabla_X A_E^*)Y,Z) = (\nabla_X B)(Y,Z) \text{ and } g((\nabla_X A_N)Y,Z) = (\nabla_X C)(Y,Z).\]

**Proof.** For any \(X, Y \in \Gamma(TM)\) and \(Z \in \Gamma(S(TM))\) we have
\[g((\nabla_X A_E^*)Y,Z) = g(\nabla_X (A_E^*Y),Z) - g(A_E^*(\nabla_X Y),Z) = X.g(A_E^*Y,Z) - g(A_E^*Y,\nabla_X Z) - B(\nabla_X Y,Z) = X.B(Y,Z) - B(Y,\nabla_X Z) - B(\nabla_X Y,Z) = (\nabla_X B)(Y,Z).\]
The second relation is obtained by similar calculation. \(\Box\)

**Definition 4.4.** A lightlike hypersurface \((M,g,S(TM))\) of a semi-Riemannian manifold \((\overline{M},\overline{g})\) is said to be locally symmetric [11] if the curvature tensor \(R\) of \(M\) satisfies the following conditions
\[g((\nabla_W R)(X,Y)Z,PT) = 0 \text{ and } g((\nabla_W R)(X,Y)Z,N) = 0,\] (46)
for any \(X, Y, Z, W, T \in \Gamma(TM)\) and \(N \in \Gamma(N(TM))\).

**Lemma 4.5.** Let \((M,g)\) be a lightlike hypersurface of an indefinite Kenmotsu space form \((\overline{M}(c),\overline{g})\) with \(\xi \in TM\). Then,
\[g((\nabla_W R)(X,Y)Z,T) = B(Y,Z)(\nabla_W C)(X,T) - B(X,Z)(\nabla_W C)(Y,T) + \{B(W,X)g(Y,T) - B(W,Y)g(X,T)\}\theta(Z) + (\nabla_W B)(Y,Z)C(X,T) - (\nabla_W B)(X,Z)C(Y,T) + B(W,Z)\{\theta(X)g(Y,T) - \theta(Y)g(X,T)\},\] (47)
and
\[
+ \{B(W, X)\theta(Y) - B(W, Y)\theta(X)\}\theta(Z),
\]

for any \(X, Y, Z, W \in \Gamma(TM), T \in \Gamma(S(TM))\) and \(N \in \Gamma(N(TM))\).

**Theorem 4.6.** Let \((M, g)\) be a lightlike hypersurface of an indefinite Kenmotsu space form \((\overline{M}(c), \overline{g})\) with \(\xi \in TM\). Then, \(M\) is locally symmetric if and only if it is totally geodesic.

**Proof.** Let \((M, g)\) be a lightlike hypersurface of an indefinite Kenmotsu space form \((\overline{M}(c), \overline{g})\) with \(\xi \in TM\). Suppose that \(M\) is locally symmetric. Then, for any \(W, Y, Z \in \Gamma(TM)\), \((\nabla_w R)(X,Y)Z = 0\). Taking \(Y = E\) and \(Z = \xi\) in (47), one obtains
\[
0 = g((\nabla_w R)(X,E)\xi, N) = B(W, X),
\]
which implies that \(M\) is totally geodesic. The converse is obvious.

In virtue of Theorem 2.3, we have the following result

**Theorem 4.7.** Let \((M, g)\) be a lightlike hypersurface of an indefinite Kenmotsu space form \((\overline{M}(c), \overline{g})\) with \(\xi \in TM\). Then, \(M\) is locally symmetric if and only if it is parallel.

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. For any \(X, Y \in \Gamma(TM)\),
\[
u([X,Y]) = B(X, \phi Y) - B(\phi X, Y).
\]
It is easy to check that the distribution \(D \perp \langle \xi \rangle\) is integrable if and only if \(B(X, \phi Y) = B(\phi X, Y), \forall X, Y \in \Gamma(TM)\).

**Lemma 4.8.** Let \((M, g, S(TM))\) be a locally symmetric lightlike hypersurface of an indefinite Kenmotsu space form \(\overline{M}(c)\) with \(\xi \in TM\). Then, for any \(X, Y, Z, T \in \Gamma(TM)\), we have,
\[
R(E,Y,Z,T) = 0, \quad R(X,E,Z,T) = 0, \quad R(X,Y,E,T) = 0.
\]

**Proof.** By Theorem 4.6, we have \(B = 0\) and the proof is completed by using relations (44).
Let \((M, g, S(TM))\) be a screen integrable lightlike hypersurface of an indefinite Kenmotsu space form \(\overline{M}(\xi)\) with \(\xi \in TM\). Using Gauss and Weingarten equations, we have,

\[
R(X,Y)Z = R^*(X,Y)Z + C(X,Z)A^*_E Y - C(Y,Z)A^*_E X \\
+ \{(\nabla_X C)(Y,Z) - (\nabla_Y C)(X,Z) + \tau(Y)C(X,Z) \\
- \tau(X)C(Y,Z)\}E, \quad \forall X,Y,Z \in \Gamma(TM'),
\]

where \((\nabla_X C)(Y,Z) = X.C(Y,Z) - C(\nabla_X Y,Z) - C(Y,\nabla_X Z)\). By covariant derivative, we have any \(W, Y, T \in \Gamma(TM')\) and by virtue of Lemma 4.3, that

\[
g((\nabla_W A^*)Y,T) = (\nabla_W B)(Y,T).
\]

If \(M\) is locally symmetric, then, using Theorem 4.6, \(B = 0\). By Lemma 4.8, \(g((\nabla_W R^*)(X,Y)Z,T) = 0\), that is \(M'\) is locally symmetric in \(\overline{M}\). Therefore,

**Theorem 4.9.** Let \((M, g, S(TM))\) be a screen integrable lightlike hypersurface of an indefinite Kenmotsu space form \(\overline{M}(\xi)\) with \(\xi \in TM\). If \(M\) is locally symmetric, then any leaf \(M'\) of \(S(TM)\) immersed in \(\overline{M}\) as a non-degenerate submanifold is locally symmetric.

Note that the locally symmetry has an integrability condition, namely, the semi-symmetry. Now, we deal with semi-symmetric lightlike hypersurfaces in indefinite Kenmotsu spaces form, tangent to the structure vector field \(\xi\).

**Definition 4.10.** A lightlike hypersurface \(M\) of a semi-Riemannian manifold \(\overline{M}\) is said to be semi-symmetric if the following condition is satisfied ([2])

\[
(R(W_1, W_2) \cdot R)(X,Y,Z,T) = 0, \quad \forall W_1, W_2, X, Y, Z, T \in \Gamma(TM),
\]

where \(R\) is the induced Riemann curvature on \(M\).

This is equivalent to

\[-R(R(W_1, W_2)X,Y,Z,T) - ... - R(X,Y,Z,R(W_1, W_2)T) = 0.
\]

In general the condition (53) is not equivalent to \((R(W_1, W_2) \cdot R)(X,Y)Z = 0\) like in the non-degenerate case. Indeed, by direct calculation we have, for any \(W_1, W_2, X, Y, Z, T \in \Gamma(TM)\),

\[
(R(W_1, W_2) \cdot R)(X,Y,Z,T) = g((R(W_1, W_2) \cdot R)(X,Y)Z,T) \\
+ (R(W_1, W_2) \cdot g)(R(X,Y)Z,T).
\]

Next, we investigate the effect of semi-symmetry condition on geometry of lightlike hypersurfaces in an indefinite Kenmotsu space form. We state the following theorem.
Theorem 4.11. [10] Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Then $M$ is semi-symmetric if and only if it is totally geodesic.

In virtue of Theorem 4.6 and Theorem 4.11, we have the following result.

Theorem 4.12. [10] Let $M$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Then $M$ is locally symmetric if and only if $M$ is semi-symmetric.

A lightlike submanifold $M$ of a semi-Riemannian manifold $\overline{M}$ is said to be Ricci semi-symmetric if the following condition is satisfied ([6])

$$(R(W_1, W_2) \cdot Ric)(X, Y) = 0, \ \forall W_1, W_2, X, Y \in \Gamma(TM),$$

(55)

where $R$ and $Ric$ are induced Riemannian curvature and Ricci tensor on $M$, respectively. The latter condition is equivalent to

$$-Ric((R(W_1, W_2)X, Y) - Ric(X, (R(W_1, W_2)Y) = 0.$$ 

In the following theorems we quote the results found by Massamba in [10] which show the effect of Ricci semi-symmetric condition on the geometry of lightlike hypersurfaces of an indefinite Kenmotsu space form.

Theorem 4.13. [10] Let $M$ be a Ricci semi-symmetric lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Then either $M$ is totally geodesic or $Ric(E, A_N E) = 0$.

Using the results above, we have the following.

Theorem 4.14. Let $(M, g, S(TM))$ be a special weakly Ricci symmetric screen locally (or globally) conformal (or $\eta$-Einstein or Einstein) lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)(n > 1)$ with $\xi \in TM$ such that $S(TM)$ is parallel. Then, the following assertions hold.

(i) $M$ is locally symmetric,

(ii) $M$ is semi-symmetric.

Moreover, if $Ric(E, A_N E) = 0$, then $M$ is Ricci semi-symmetric.

It is well known that the screen distribution of a manifold $M$ is not unique but the local second fundamental form $B$ of $M$ is independent of the choice of the screen distribution therefore all of the results which depends only on $B$ are independent of the choice of the screen distribution and are stable with respect to the vector bundles $(S(TM), S(TM^1))$ and $N(TM)$. It was shown in [1, page 87] that a change from a screen distribution $S(TM)$ to another $S(TM)'$, for $C$
and $C'$ the local second fundamental forms of $S(TM)$ and $S(TM)'$, respectively the above results are independent of the screen distribution if and only if 
\[ \omega(\nabla_X PY + B(X,Y)W) = 0, \]
where $\omega$ the dual of 1-form of $W = \sum_{i=1}^{2n-1} c_i W_i$, called a characteristic vector field of the screen change, with respect to the induced metric $g$ of $M$, that is, \( \omega(X) = g(X,W), \quad \forall X \in \Gamma(TM). \) And $P$ a projection of $TM$ on $S(TM)$ with respect to the orthogonal decomposition of $TM$. Here $c_i$ are smooth functions on $U$.

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**References**


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